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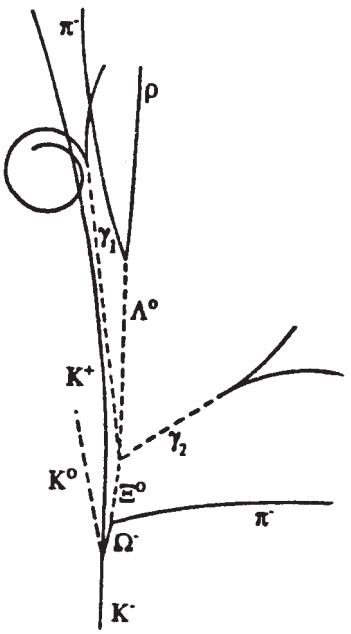

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SOME THOUGHTS ON SPECIAL RELATIVITY

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Abstract

Here an introduction to Wesley's neomechanics is presented. It is shown to produce some of the same results as Special Relativity but without both the mathematical and philosophical basis of that subject. As with other work in which results associated with General Relativity are obtained without recourse to the fundamental bases of that subject, so here too the pre-eminent place afforded Special Relativity in modern science is called into question. The opportunity is taken to extend Wesley's ideas to the case where the mass of the body when at rest is not constant.

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Introduction.

Qualms concerning at least some aspects of the Special and General Theories of Relativity seem to have existed ever since both broke into modern science. However, in more recent years, some very real queries have been raised concerning the true place those theories should occupy in science. For example, as far as General Relativity is concerned, three crucial tests are normally discussed in most, if not all, textbooks on the subject. However, it has been shown[1] that, as far as the test associated with the gravitational red shift is concerned, although the final result draws on modern results due to Planck and Poincaré, it is fundamentally deduced from notions of Newtonian mechanics alone. Further, in a 2005[2], Lavenda showed that the deflection of light and the advance of the perihelion of Mercury may be treated as diffraction phenomena on the basis of Fermat's principle and the modification of a Bessel function in the short wavelength limit. He has also shown, in the same article, that the problem of time delay in radar echoes may be treated by the same means. Hence, where does that leave the General Theory of Relativity? It must be admitted that that theory does seem to lead to correct results but it is now clear that those same results may be obtained by other means which do not involve any of the ideas of General Relativity. This certainly denies the General Theory of Relativity the pre-eminent position it has seemingly occupied in physics for many years.

However, this says nothing of the Special Theory of Relativity and possibly even more qualms have been harboured about that theory for many years and have led to the almost complete ostracising of such as Herbert Dingle. As far as Dingle's particular case is concerned, the complete history is chronicled in detail in the book *Science at the Crossroads*[3], which may be found on the internet. It is interesting, therefore, that manipulations by J. P. Wesley[4] seem to have gone unnoticed.

Although the main object is to make more people acquainted with the ideas of Wesley, the opportunity has been taken to extend that gentleman's results to the case when the mass of the body when at rest is not a constant but may vary with time.

Wesley's Neomechanics.

Although rarely mentioned, it is a long known fact that there is a strong relationship linking mass and energy or, in other words, the idea of

mass/energy equivalence was known and used long before Einstein's 1905 article in which, to many, it was introduced into science for the first time. In the 1800's, the topic was investigated at length and the notion that the two were linked via a coefficient of the order of c^2 was established by electrodynamic considerations. One pre-1905 reference to this is to be found in J. J. Thomson's book *Electricity and Matter*, which is a printing of a series of lectures Thomson gave at Yale University in May 1903, but the history of the relation goes back much further in time. Although there was some doubt for a time over the exact coefficient linking mass and energy, it became apparent that that coefficient was c^2 and that has been verified experimentally to a high degree of accuracy. Wesley's so-called neo-mechanics is based on this vitally important principle of mass/energy equivalence as expressed in

$$E = mc^2 \qquad (1)$$

and has absolutely no connection with the space-time of Special Relativity. For those unacquainted with Wesley's work, it is worth examining his deduction.

Where Wesley diverges from traditional Newtonian mechanics is a result of his noting that, since mass/energy equivalence is an established fact, if this applies to any form of energy, it follows that there must be a mass equivalent for kinetic energy. This fact has to be included, therefore, in traditional Newtonian mechanics as a modification. Consider a body at rest whose measured mass is m. Suppose this same body then moves and when in motion, possesses a kinetic energy T then, the mass equivalent is T/c^2 and, therefore, the total momentum of the body is given by

$$\boldsymbol{p} = (m + T/c^2)\boldsymbol{v}$$

From Newton's Second Law, it follows that the force acting, P, is given by

$$\boldsymbol{P} = \frac{d}{dt}\left(m + {}^{T}/_{c^2}\right)\boldsymbol{v}$$

Since the rate of working of the force equals the rate of increase of the kinetic energy, it follows that

$$\frac{dT}{dt} = \boldsymbol{v}.\frac{d\boldsymbol{P}}{dt} = \boldsymbol{v}.\frac{d}{dt}\left(m + {}^{T}/_{c^2}\right)\boldsymbol{v}$$

Next note that, if $\gamma = \left(1 - {}^{v^2}/_{c^2}\right)^{-1/2}$,

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2}\boldsymbol{v}.\frac{d\boldsymbol{v}}{dt}$$

and so

$$\frac{dT}{dt} = \boldsymbol{v}.\frac{d}{dt}\left(m + {}^{T}/_{c^2}\right)\boldsymbol{v} = \left(m + {}^{T}/_{c^2}\right)\boldsymbol{v}.\frac{d\boldsymbol{v}}{dt} + (\boldsymbol{v}.\boldsymbol{v})\frac{1}{c^2}\frac{dT}{dt}$$
$$= \left(m + {}^{T}/_{c^2}\right)\frac{c^2}{\gamma^3}\frac{d\gamma}{dt} + \frac{v^2}{c^2}\frac{dT}{dt}$$

Rearranging leads to

$$\frac{dT}{dt} = (mc^2 + T)\frac{1}{\gamma}\frac{d\gamma}{dt}$$

or

$$\frac{1}{(mc^2 + T)}\frac{dT}{dt} = \frac{1}{\gamma}\frac{d\gamma}{dt}$$

Integrating both sides with respect to t and noting that $T = 0$ when $v = 0$ leads to

$$T = mc^2(\gamma - 1),$$

a result normally associated with the Special Theory of Relativity.

A modification of Wesley's neomechanics.

In the above m, the mass of the body when at rest, is tacitly assumed to be constant. However, it is not unreasonable to wonder what modifications would need to be made to the above if m was not constant; it can be imagined to alter with time for example. Consequently, assume m to be a function of time. Under these circumstances, the equation for dT/dt above becomes

$$\frac{dT}{dt} = \boldsymbol{v}.\frac{d}{dt}\left(m + {}^{T}/_{c^2}\right)\boldsymbol{v} = \left(m + {}^{T}/_{c^2}\right)\boldsymbol{v}.\frac{d\boldsymbol{v}}{dt} + (\boldsymbol{v}.\boldsymbol{v})\left(\frac{dm}{dt} + \frac{1}{c^2}\frac{dT}{dt}\right)$$
$$= \left(m + {}^{T}/_{c^2}\right)\frac{c^2}{\gamma^3}\frac{d\gamma}{dt} + \frac{v^2}{c^2}\frac{dT}{dt} + v^2\frac{dm}{dt},$$

that is,

$$\frac{dT}{dt} = \frac{(mc^2 + T)}{\gamma}\frac{d\gamma}{dt} + v^2\gamma^2\frac{dm}{dt}. \qquad (2)$$

But,

$$\frac{1}{\gamma}\left(\frac{dT}{dt} - \frac{T}{\gamma}\frac{d\gamma}{dt}\right) = \frac{1}{\gamma^2}\left(\gamma\frac{dT}{dt} - T\frac{d\gamma}{dt}\right) = \frac{d}{dt}\left(\frac{T}{\gamma}\right).$$

Hence, the above equation (2) becomes

$$\frac{d}{dt}\left(\frac{T}{\gamma}\right) = \frac{mc^2}{\gamma^2}\frac{d\gamma}{dt} + v^2\gamma\frac{dm}{dt}.$$

Then, integrating throughout with respect to t, remembering that m is a function of t, leads to

$$\frac{T}{\gamma} = c^2\int\frac{m}{\gamma^2}d\gamma + \int v^2\gamma dm + const.$$

$$= c^2\left[\int\frac{1}{\gamma}dm - \int d\left(\frac{m}{\gamma}\right)\right] + \int v^2\gamma dm + const.$$

$$= \int\left(\frac{c^2}{\gamma} + v^2\gamma\right)dm - \frac{mc^2}{\gamma} + const.$$

$$= -\frac{mc^2}{\gamma} + \int\frac{(c^2 + v^2\gamma^2)}{\gamma}dm + const.$$

Substituting for γ and simplifying leads to

$$\frac{T}{\gamma} = -\frac{mc^2}{\gamma} + \int c^2\gamma dm + const.$$

Once again $T = 0$ when $v = 0$ and so the value of the constant is given by

$$mc^2 - \left[\int c^2\gamma dm\right]_{v=0}$$

Hence, the final result may be written symbolically as

$$T = mc^2(\gamma - 1) + \int_{v=0}^{v} c^2\gamma dm.$$

Comments.

It should be noted that none of the above derivation in the section purely devoted to an introduction to Wesley's neomechanics is new; it may be found, albeit in a somewhat abbreviated form, in Wesley's book[4]. However, it does appear that this work is not readily available and it seems it is certainly largely unknown in scientific circles. However, although the introduction of the usual Special Relativity factor γ is crucial and it is not clear how or why that factor is introduced, it is seen that the expression for the kinetic energy normally associated with Special Relativity is derived without recourse to the methods or philosophy of that subject. This is basically the starting point for Wesley's so-called neo-mechanics. True; it does have some of the features of Special Relativity but that is where any similarity ends. Neo-mechanics has a totally

different base from Relativity and is without the awkward results which accompany that popularly accepted theory. Those awkward results of the mathematical theory which have caused so many problems over the years and have led to literally reams of paper devoted to numerous publications, all attempting to justify a physical interpretation for what are purely mathematical results, are missing. However, the useful, physically justified results remain.

Yet again then, results associated in most peoples' minds exclusively with relativity are found to be derivable by alternative methods. Just as the results alluded to in the introduction raise grave questions about the place afforded General Relativity, so this discussion here, highlighting ideas of Wesley, raises similarly serious questions about the pre-eminent place afforded Special Relativity in modern science. In the case of General Relativity, no-one was claiming that subject incorrect; they were simply pointing out that a lot, maybe all, of the results on which the various crucial tests are based were obtainable by alternative methods which relied on mathematical techniques which had been in place for many years. They did not draw on, for example, results of Riemannian Geometry. In that case though, if it became clear that a result was obtainable only via the new methods that might indicate a need for caution when using the said result until an alternative derivation is found. The position in Special Relativity is, however, possibly more serious because, in that case, there are several results – to which reference has been made already – which cause grave problems when any attempt is made to fit a seemingly feasible physical explanation to them. With Wesley's approach, the equations of the Lorentz transformation simply do not appear and that in itself removes many awkward aspects of the currently favoured theory. However, the truly crucial result, equation (1) above, which should not really be regarded as the exclusive property of Einstein and his Special Relativity, remains and forms the cornerstone for this new approach.

In the section devoted to a modification of Wesley's work, the mass of the body when at rest is assumed to be non-constant and to be, in fact, a function of time. It seems it cannot depend on anything else since all effects due to subsequent motion are contained in the kinetic energy. However, one can envisage effects which might cause the mass of a body while at rest to alter over time and so this slight modification is, it seems, worth considering. Once again though, it should be emphasised that, although the familiar factor γ is very much in evidence in all the expressions, there has been absolutely no recourse to the methods or philosophy normally associated with Special Relativity.

The crucial point to be drawn from all this is that, as with the examples mentioned associated with General Relativity, a practically important result usually regarded as being solely a consequence of Special Relativity has been obtained without utilising any of the basics of that subject. This does not even hint at Special Relativity being incorrect but does show that some of its results, at least, are obtainable by other means. This, in turn, casts doubt on the position and usefulness of those results which have led to the awkward paradoxes that have caused so much intellectual effort to be spent on attempting to explain them; in fact, to find physical explanations for results which, quite simply, may not admit such explanations since the results are themselves merely non-physical consequences of the particular mathematical techniques initially used to investigate some genuine physical phenomena.

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RETHINKING THE FORMAL METHODOLOGY (I): WAVE-VORTEX ESSENCE OF THE SUBSTANCE

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Abstract

An approach/methodology proposed to basic problems, alternative to standard formalism. The opportunity of representation of the quantum phenomena in causality principle, on the wave-field common nature of the substance is shown. Elementary particles physical essence and types of interactions interpreted within wave-field peculiarities. The problems with de Broglie wave and particles' double slit interference discussed. Physical models of basic hadrons, their internal structure and static fields' configurations proposed. The values of mass, spin, magnetic moments of *n*, *p* hadrons defined within modeling. Causal interpretation to β decay presented. The tremendous penetrating peculiarity of neutrino discussed. Structural schemas to ${}_2^4H_e$, ${}_6^{12}C$ nucleons are proposed.

Keywords: Standard Model, Proton, Neutron, Nuclear Structure

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1. Introduction

1. Critical Remarks

Ancient philosophers have put forward the "Main Issue" with comprehensive formulation, "From what kinds of primordial substances and by which principles exists the material world?" The statement of the problem has essentially changed by time. It sounds as follows in the language of contemporary physics: "By which minimal quantity of natural constants and on the basis of what equations is it possible to describe all kinds of physical phenomena?" There are no principal contradictions between two statements. Meanwhile, these are significantly different by meaning and in content that we shall examine in attempts to comprehend where from arises present unprecedented complexities in disputable sections of physics. Author is inclined to see the explanation of current difficulties within previous serious misconceptions at the basic level. He proposes some retreat and rethinking of a traversed path, by simplest reasoning: such audit may give a chance only, and no harm, as it is impossible to lose what is known. We cannot exclude possible mistakes of deserved pioneers and ignore the necessity of revising the past way sometimes, which is natural at any research. We start with some remarks concerning to known ***"Copenhagen's Interpretation"***, introduced at the beginning of past century as a revolutionary approach, in context of comparison. We shall look at some points only since the issue was widely discussed among prominent luminaries long ago. To judge the significance of introduced methodology let us begin with known facts:

1. A key postulate of quantum mechanics (QM), ***a probable/statistical interpretation of Schrödinger's equation*** (SE), ***was accepted with majority of opinions*** (i.e. politically and not on the objective arguments!)
Hence, it is right to look at the mentioned interpretation as situational, "ad-hoc" approach only, i.e. as a subject, which needs further clarification, and not as a doubtless "basic principle". The mentioned interpretation is mainly justified with its certain productivity. With careful observation, we can guess that it is an obvious misunderstanding to attribute results of any theory to a conditional name, or to its declarative interpretation. Nevertheless, ***the productivity of*** QM ***may evidence its quantitative accordance to reality only, and not its intuitive/verbal interpretation, which can be any!*** The issue gets trivial solution with logical viewpoint. Taking into consideration that SE is based on the Hamilton/Laplace

operators that are generalized, causally defined relations (differential, in math meaning), we can assert; ***corresponding causal interpretation of*** SE ***and quantum phenomena should exist*** (the possibility of imaginary representation of quantum phenomena as cause-effect, consecutive chains). This type of description totally differs from nowadays-accepted formal methodology based on the abstract, math reasoning. It corresponds to human's natural ability of thinking, by means of images and actions that can make physics a comprehensive-realistic science. Here we need to remember that: ***the*** SE ***is deduced with generalization of de Broglie wave-particle dual properties and optics' geometry, by cause-effect, quantitative reasoning. Hence, it is right to look at the "workability" of*** SE ***as an evidence of correctness of the way and principles of its deduction*** (and not to its arbitrary interpretation!) In Schrödinger's deep confidence, SE should have corresponding causal representation that he resolutely attempted to bring to colleagues' awareness. However, majority decided how to interpret it and the author was accused for "naive-realistic" ideas (!) **[1]**. SE ***definitely means to look at the localized elementary particle as a standing wave packet, by logic of its deduction***. Therefore, realistic thinkers have tried to prove the concept. The approach has been recognized as "wrong" due to not being successful to prove it mathematically. The statistical/probable interpretation of QM has been introduced then, with the formal recipes and instructions of new methodology. We shall emphasize following obvious-subjective action in this crucial event: ***defining something as "impossible/unacceptable" since we did not manage to do it (!)*** Such unprecedented decision has met a deep protest of known coryphées, who continued their efforts of building a realistic science, remaining in dramatic minority. With Einstein's confidence, ***the division of mater and el. charge from field as different kinds of realities, after establishment of*** $W=mc^2$ ***seems unnatural*** **[2]**.

He has insistently attempted to complete his ***Unified Field's Theory*** working in full isolation from the community (for the last 30 yrs of his life!) Meantime, de Broglie has managed to explain one key quantum phenomenon within cause-effect principle: a two-slight interference of particles (***de Broglie-Bohm theory***). Mentioned solution proved the rightness of Schrödinger's viewpoint. ***De Broglie-Bohm theory shows the principal possibility of cause-effect interpretation of the quantum phenomena***. Nevertheless, such important result remains yet "invisible" in community. A new ideology was accepted there, as having no

alternative, since certain results had been achieved already and it gave a wide opportunity for math manipulations, in author's view. Meantime,

The cause-effect interpretation of quantum phenomena may open an opportunity to build physics on general principles and to complete it on unique conceptual basis. The formal methodology ignores/excludes the mentioned opportunity.

2. QM ***methodology is not enough adequate quantitatively***

Let us demonstrate first the subjective/uncritical character of applying operations in below example. The wave function in stationary SE for freely moving particle (external field's potential *U=0,* a particle moves along *x,* with speed and energy; *v=const, w=mv²/2*) given as:

$$\psi = A\exp[-\frac{i}{\hbar}\sqrt{2mW}x] + B\exp[\frac{i}{\hbar}\sqrt{2mW}x] \qquad (1.1)$$

Here *A, B* are certain constants, remaining in QM as obscure. The complex conjugate function is introduced in next step:

$$\psi\psi^* = \left|\psi^2\right| = \left|A^2\right| \qquad (1.2)$$

However, QM ***doesn't give an answer/argument; where from and why*** ψ^* ***appears there***? It means an unjustified operation takes place here (even, from formal viewpoint) which just looks as necessary (?) to link the solution (1.1) with the eq. (1.2). Moreover, a new concept is introduced and a new supposition is accepted then:

$$\left|A^2\right| = \frac{dw}{dv} = \rho \qquad (1.3)$$

Where: ρ is declared as the "density of probability" (the probability of finding the particle in the elementary volume). Below expression is accepted then as the measureless "unit of probability":

$$\int_v \left|\psi^2\right| dv = \int_v \left|A^2\right| dv \equiv 1 \qquad (1.4)$$

We would notice that measureless numbers arise in descriptions of real phenomena as ratios of physical values to basic ones, having the same kinds. Passage from real/physical to relative values assumes initial definition of their kinds and basic measures, which is possible if we are clearly aware of the nature of studied objects and physical essence of phenomena. The introduced relative unit (1.4) "suspends in the air"; it stays as cognitively dark category, since QM initially speaks nothing about physical nature of wave function. Therefore, the necessity of "choosing" its interpretation/affiliation, as well as

for definition of basic measures and borders of its existence/action arises further (which was done in QM by votes, arbitrary actions and unexplainable instructions, as we saw above).

3. A QM *methodology has internal inconsistencies*

Let us examine following example. The quantum objects may be presented as "particles" or "waves", and not as "waves and particles" at the same time, with interpretation of de Broglie's duality principle (DP), accepted in QM. The freely moving particle is described there with wave function (1.1) in accordance to it. In fact, we deal with a "particle" that generates a "wave" at its movement, which are indivisible from each other. The QM recognizes the mentioned indivisibility as well. However, it becomes verbal/psychological declaration only, since the QM allows presenting/describing one kind of object, and not two types together ("wave" and "particle") as it is in reality. I.e. ***the*** QM ***accepts two kinds of objects, existing together at the same time, and it allows describing one of them only*** (?) Reader will see further that the mentioned arbitrary rule causes serious misconceptions and confusions. We would bring to readers' attention also that, in fact, DP becomes disturbed in quantum el. dynamics (QED). The moving particle is presented there as in permanent interaction with the virtual photon, as the process of alternating radiation/absorption of it by particle. Thus, the described objects become two kinds, vs. QM' rule. We will show further that success and productivity of QED are particularly related to its silent departure from mentioned unexplainable recipe.

4. The "Point-like" representation of objects contradicts to reality

The mentioned point is important to comprehend as one of the main problems of formal methodology, as well as the priority of suggested approach. The QM has developed in analogy to Newtonian physics; therefore, some of idealized concepts have uncritically passed into it. The issue is about the concept of "material point" that is principally not compatible to reality. However, it is used in classical physics with strong restriction: the own sizes of moving objects should be insignificant compared to distances in studied phenomena. We deal with restricted distances in microcosm not having enough criteria that objects of study are too small in relation to these. Therefore, initial acceptance of objects as "insignificantly small" is unjustified (we do not have enough experimental knowledge of them as in classical physics). The second point is more important. We lose any opportunity to comprehend the physical essence

of quantum objects by declaring them as "point-like". We just have closed for ourselves the way to their realistic description, initially accepting them as "something that is not subject to representation with sizes, images and actions"; by the same, the abstract math reasoning only remains as the single tool of analysis.

5. The statistical interpretation of phenomena contradicts to facts

We will look at one obvious example only, from many similar. The statistical distribution of atomic electrons is accepted in QM as "clouds of probability" within above interpretation of wave function (*1.1.2.*) The emission of photons is interpreted as consequence of electrons' passage from one probable/average parameter into the other. It follows; the emitted single photon should have certain deviation in parameters, which means a spectral line of single photon will have no exact place on the screen but an average. Hence, a spectral picture should lose sharpness with decreasing intensity close to fluctuation level. As known however, a spectral picture does not depend on intensity, which shows that emitted photons have strongly determined parameters. An obvious question comes out then: how does exact/determined photon arise from "cloud of probability" (?) However, reader should know that natural questions are not subject of QM, by definition. Therefore, the answer to such questions remains only one, "our formulas show it". It actually means, "God has made our world as it is" because our formulas are constructed as artificial models of reality, and not as natural derivatives from basic concepts, with accepted methodology. Thus, we have the right to assert; the probable/statistical interpretation of phenomena and rejection of cause-effect principles actually prohibit the cognitive study of the subjects.

6. The conceptual-cognitive representation of phenomena is absent in formal methodology. The "composition/construction" of the quantitative descriptions are accepted as the final task of the theory

R. Feynman has truly formulated the mentioned point **[3]** that we have seen in previous content as well. Depicted reality means "to harness a cart ahead the horse", from logical viewpoint. It explains why realistic thinkers could not accept the new approach, and how they appeared to be in deeper opposition. Einstein demanded at his time, ***to build physical theories on conceptual basis,*** and, ***to use concepts connected to reality*** **[2, 4]**. It simply means to grasp the essence of studied phenomena initially and not to input hypothetical objects at each difficult case (see: ***"Occam's Razor"***). However, majority has seen unlawfulness of these demands, and

Einstein got the "corresponding" stamp as well (see: ***Einstein's operationalism***). Modern theories have been developed then as a specific genre of creation, having no borderlines between reality and fiction. It has brought to a loss of objective criteria as well (estimation of work's significance on the "indisputable" opinions, quantity of citations, by "brand" etc, and not on the workable results corresponding to reality, i.e. politically and not essentially). It has caused hard polarization between main types of scientists: "thinkers by own brains"- "intransigent rebels" as a rule, and respectable scientists, "followers of holy instructions" in majority, as natural. Professor ***R. Santilli*** impressively depicts nowadays reality **[5]**. Numerous unclear approaches and unsubstantiated doctrines have been introduced into disputable sections as consequence. Described way of building "Basic Science" is unable to withstand elementary logical criteria. It has been decided then to look at the logical requirements as a kind of philosophical category that are "unseemly and trivial" for the leading science (which can be even "harmful"!)[1] Most scientists, trained in such spirit, already seriously believe in the possibility to explain at last, "How God created the world" within symbols and math formulas only, excluding natural language and thought. The abstract mathematics (i.e. our "work-tool"!) has become priority in result, which may even "show us" the necessary direction of research. It is difficult therefore to imagine how the desired "Final Theory" will look. Judging from dominating ***Standard Model*** it will be a big collection of sophisticated equations completed with numerous "calibrating", "normalizing" functions and factors, experimentally and empirically introduced. A limited quantity of individuals will be able "to understand" it (mastering the tremendous volumes!) We also cannot guess what purpose such theory may serve to (other than demonstrating the "merits" of scientists). It will be just craftiness to talk about "cognitive significance", since similar categories are accepted as "unscientific" at the beginning. Reader can judge that "Final Theory" will be useless also from technological viewpoint, by simplest reasons. ***The absence of common view and actual inability of researchers to formulate the purpose of their job are the obvious/alarming symptoms of deep confusion of modern physics***, in

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[1] There is no exaggeration here. The physical theory accepts the "math modeling of reality" without any conceptual paradigm, within present ideology (see, for example, ***Kuhn, T.S., "The Structure of Scientific Revolutions"***, University of Chicago Press, 1962. ISBN 13: 9780226458083)

author's opinion. The formal methodology actually has brought to certain impressing results at the beginning (QED etc.) Nevertheless, huge technical problems and gnoseological fog quickly have arisen against physicists. Several speculative-formal theories and accompanying tremendous experiments, with unclear goals, periodically have been developed afterwards (as ongoing LHC project[2] etc). Thus, modern physics actually has gone in trivial way of "test-error" for unlimited math exercises, and not in any consistent direction. Much of objective criticism and unanswered questions permanently proceed from large group of realistic thinkers, reflecting the present crisis. Numerous "reviewers" and "moderators" are engaged on "neutralizing" these at different levels, applying ready stamps, under noble purpose of "protecting the truth from heresy!" As we see, ***the formal methodology actually welcomes uncritical compositions and it resolutely prohibits natural way of thinking*** (!) It corresponds to "amputation of brains", by Einstein's definition, which may reliably stop any progress of science. Is the depicted reality a result of simple misunderstanding, manifestation of group's ambition in a worse scenario, or more serious circumstances are covered here? We do not undertake to speculate on this direction, recalling just that similar situations have taken place in science history enough often, by explainable reasons. In author's view, the current reality of modern physics is far incompatible with scientific spirit and undeclared honesty. It may call only a regret of true thinkers who can hope on future in such case. ***The present crisis of physics requires general revision of methodology, the statements of problems and accepted criteria of significance on disputable subjects***. We attempt to do it in the scope of this work, particularly.

2. Compatibility of the approach to existing theories

The proposed approach is an attempt to present the picture of reality of disputable subjects in possible-complete form, in author's view and in his ability. There is no initial intent to prove or reject any existing approach concerning the issue. It is right to look at this work in context of critical overview of several existing theories based on different methodological principles. ***The main criterion of selection was the compliance of results to established facts*** (leaving aside various interpretations and large

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[2] The opening of "Higgs Boson" has been announced at present. Reader can judge the significance of the event with mastering the article.

terminology accompanying these). The approach demands revision and rejection of misinterpretations, unreasonable instructions, intuitively accepted beliefs/doctrines, and not the existing facts, actual results. We can assert therefore that it satisfies to "***Correspondence Principle***" in the extent by which comparable works conform to established facts. Proposed changes mainly have cognitive-psychological character that always has played painful and huge-resistive role. The problem aggravates more as the proposed approach has already been examined. It has got a final verdict "wrong" and main ideologists have decided, "The page is closed!" (*1.1.1.*) However, similar situations also happened in science history. We hope therefore, that presented work may be perceived in right meaning and significance with time. ***Disputable areas of physics may get the unique conceptual basis, which gives real opportunity to separate valuable approaches and results from existing plenty of unnecessary***, in author's confidence.

3. Basic principles

We present key points of the approach with some substantiation:

1. Quant of Electromagnetic Field (QEF) ***is a unique base of substance***

a). Representations of the photon and localized particle (electron) with their known properties as kinds of QEF manifestations presented in **[6]**
We present some additional arguments confirming the concept.

b). The existence of unique couple of universal constants c, h confirms the concept. There are no experimental evidences to existence of other constants with similar significance (let's say c_1, $c_2 ... c_n$, h_1, $h_2 .... h_n$). There are no principal arguments excluding their existence as well. Hence, their uniqueness in fact points at the unique nature of the basic substance.

c). Similar physical characters and properties of different elementary particles and their equality (such as existence and equality of spin, el. charge) obviously evidence the unique nature of all known kinds of particles. Otherwise, the mentioned similarity/equality becomes unclaimed, unexplainable coincidence (that actually seems in the Standard Model!)

d). Known possibility of mutual transformations of all kinds of elementary particles (by accepted terminology) into each other (within conservation laws) directly confirms their unique essence (independent from our ability to prove it theoretically; as mentioned, the math proofs may look more significant than existing facts in formal methodology!)

e). The electromagnetic, wave-field nature of basic substance derives from Einstein's $W=mc^2$, $W=h\nu$ equations (to be discussed). Many of researchers have been working on this direction (as example: **[7, 8]**.) We will refer to Feynman's phenomenal intuition as well on the issue. He has seen identical results in different kinds of formal theories as weighty evidence to uniqueness of basic laws and principles of the nature **[3, 9]**

We present a brief description of QEF as a candidate of basic substance:

2. The QEF ***has a wave-vortex, dynamic nature, presenting itself a circularly polarized, restricted wave flow*** (quantum wave packet)

The existence of spin for known kinds of elementary particles, without exception, evidences the vortex-dynamical nature of QEF. The interpretations of spin for the electron and photon within named concept is presented in **[6]**

3. The QEF ***manifests in two possible physical forms:***

a). As propagating stable quantum wave packet (photon, γ - quanta)

b). As localized/standing wave-vortex, unstable mainly, and stable in few special cases, showing general and individual peculiarities (kinds of localized elementary particles/antiparticles, stable/unstable)

4. The Mass and pseudo static fields ("charges") ***arise in localized*** QEF ***aftermaths of interferential redistribution of wave energy***

Origins of mass, spin, static electric/magnetic fields ("charges") for the electron are interpreted within the concept in **[6]**

5. All kinds of QEF ***interactions have electromagnetic nature***

We represent their possible kinds of manifestations as below:

a). Mutual interactions of none localized QEFs (photons)

b). Interactions of none localized QEFs ***with pseudo static fields of localized*** QEFs (It causes absorption/emission of photons etc)

c). Interaction of localized QEF***s within their pseudo static fields*** (These are analogues of Coulomb and Lorentz's forces defining atomic orbital structures, binding energy in atomic nucleons, nuclear structures, etc)

d). Interaction of localized QEF***s in the range of their mutual coverage*** (mostly corresponding to accepted "weak interaction")

e). Internal interactions in QEF (causes propagation of photon, origin of mass, "charges", mag. momentum of localized particles, stability and decay of particles, phenomenon of gravity, breaking of symmetry (?) These are subjects of modern formal theories, such as QED, Standard Model etc.)

f). Simultaneous actions of above-mentioned kinds of interactions mostly take place with QEFs. The unique, electromagnetic nature of forces and interactions in microcosm (excluding gravity) is shown within

formal-quantitative reasoning (***Glashow, Weinberg, Salaam, Swinger*** and others). Numerous theorists recognize it, having different views at the other problems. The attempts to create a unified theory continue.

4. Methodology

Applied methodology significantly differs from formal, accepted at disputable sections. The hypertrophied role and some mystification of the mathematics in the research have been shown in previous content. We will present a brief explanatory to definitions of "realistic methodology" vs. "formal", for perception of work by readers having standard education.

We understand the "math reasoning" as quantitative logic tool, which is based on abstract conservation laws. We consider it as an important component of realistic methodology that includes other inevitable/necessary tools as well, such as the "induction", "deduction", "supposition", "syllogistic compositions/conclusions", etc. ***As a "tool" and "component", the math reasoning cannot have priority and guide us solely*** (as any other tool!) As known, the quantitative analysis often yields to variety of results that incompatible with the physical reality (as any "tool" it may have unclaimed applications!) Hence, ***in "realistic methodology" the "math reasoning" must work within comparison to other "tools" of study, under mutual control and restrictions*** (it means, we should comprehend the subjects of study and counting in principle).

We present key points of approach as below:

1. We have looked at the established facts as indisputable basic arguments (we are forced to emphasize it, vs. formal methodology!)

2. We have accepted cause-effect relations as the basic laws of nature, describing QEF (Maxwell's quantized wave equations) **[6]**

3. The basic natural constants are considered two: *c, h,* ***reflecting dynamical and quantitative characteristics of*** QEF, ***experimentally established. Measureless constants*** π, $\alpha \approx 1/137$, ***appearing in descriptions of*** QEF, ***are conditioned by its wave-vortex, dynamic nature; these are possible to deduce at the conceptual basis*** (π known from geometry, α deduced as wavy peculiarity **[6]**)

4. All kinds of properties and peculiarities of quantum objects are conditioned and are possible to represent with four basic constants, mentioned in previous point (such particles' mass, sizes, kinds' of energy, its "charge", spin, mag. momentum etc)

5. We combine the causal/quantitative reasoning with imaginary representation of objects/actions in analyzing process

6. We exclude operations/reference with supposed objects that are not confirmed experimentally as kinds of realities (the "ether", "physical vacuum", "quarks", "gluons", "graviton", "dark energy/mater" etc)

7. We looked at the possibility of representing numerous established facts in the frame of common concept and methodology as the main evidence of reliability/significance of the presented approach

2. *Results*[3]

1. Quantized Wave Equations (QWE) *as basic laws*

We show below the possibility of causal interpretation of known key phenomena, relations and principles at the base of QWE

1. A causal representation of the Heisenberg's Uncertainties (HU)

The physical essence of HU is represented in **[6]** (chap. 2) as the consequence of "wave beating", causing ***the deviation of parameters, peculiar to a quantum wave packet.*** The origin of uncertainties in the parameters of QEF and its equivalence with the HU are interpreted there.

2. Representation $W=h\nu$ *as wave-field' energy*

The QWE is presented by (37) in **[6]**, as particular solution of Maxwell's equations, corresponding to vector representation (Figure 1.)

$$\mathbf{E}_V \approx \frac{\omega}{2\pi}\sqrt{\frac{h}{2c\sigma n}}\left[\mathbf{i}\sin\omega(t-\frac{x}{c})+\mathbf{k}\cos\omega(t-\frac{x}{c})\right]$$

$$\mathbf{E}_H \approx \frac{\omega}{2\pi}\sqrt{\frac{h}{2c\sigma n}}\left[\mathbf{i}\sin\omega(t-\frac{x}{c}-\frac{1}{\omega})+\mathbf{j}\cos\omega(t-\frac{x}{c}-\frac{1}{\omega})\right] \qquad (2.1)$$

Where: $\mathbf{E}_V$, $\mathbf{E}_H$ vectors of field's tension in mutually perpendicular planes (Fig.1), $\sigma\approx\Delta\lambda^2$ section of "ring-strings" for the energy' concentration, $\omega=2\pi c/\lambda$ cyclic frequency, $\Delta\lambda$ is the HU for wavelength: $\Delta\lambda\approx\alpha\lambda$, $n\approx1/(\eta\alpha)^3$ number of pairs of whole waves composing wave packet, $\eta=m/p$, (m, p whole positive numbers), α Fine Structure Constant. Its value deduced as ***interference redistribution constant*** (new introduction) $IRC \cong e_* \equiv a^{0.5}$

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[3]Author recommends to read **[6]** previously

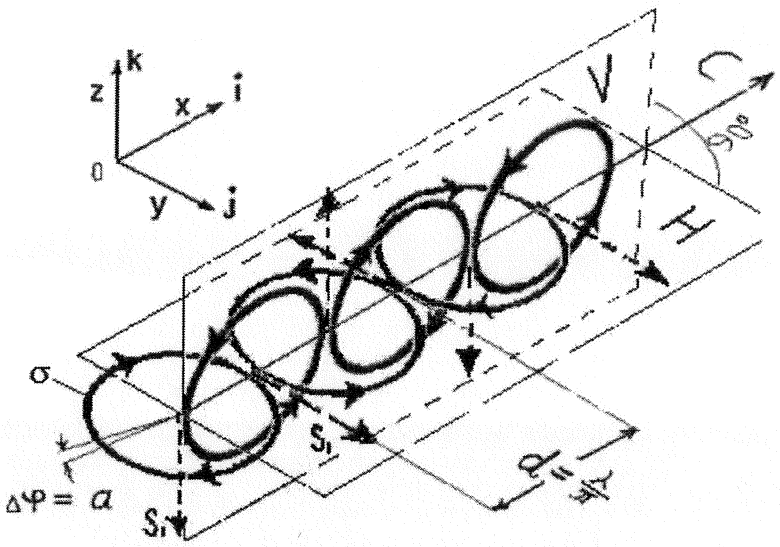

Figure 1. The Scheme of Quantum Wave-Packet

IRC defined as $a^{0.5} \cong \sin\Delta\varphi \cong 0.0854245.. \cong e_*$ (e_* elementary charge in natural units' system $\hbar = c = 1$), $\Delta\varphi$ defined in **[6]** from below equation:

$$\sum_{m=1}^{n\to\infty}\left\{\frac{1+\cos[\Delta\varphi m-\delta_\theta/(1-\delta_\theta/\Delta\varphi m)]}{(2m+1)[1-(\delta_\theta/\Delta\varphi m)^2]}\right\}^2 \approx \pi^2\tan\Delta\varphi \qquad (2.2)^4$$

Where: $\delta_\theta \approx \Delta\varphi(1-\cos\Delta\varphi m)/\pi(2m+1)$

We shall define energies' density in "ring-strings" (Fig. 1) as scalar square of $\mathbf{E}_V$, $\mathbf{E}_H$ vectors.

Considering equality of their modules, using Kronecer's symbol $\delta_{ij} = \{1(i=j),\ 0(i\neq j)$, and $sin^2A+cos^2A=1$ from (2.1) we get: $\rho \approx \langle\mathbf{E}\rangle^2 \approx \omega^2 h/8\pi^2 c\,\sigma n$. Total energy for the packet defined as:

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[4] It is important to mention that eq. (2.2) is deduced on pure cognitive basis, i.e. **on conceptual principle** exclusively, vs. numerous representations of α by different combinations of other known basic constants. Due to technical complication, the solution of eq. (2.2) has been found by direct numeric fitting for the first 10^5 numbers in the flow, adding the residual sum of $10^5 \to \infty$ numbers, which are defined by asymptotic approximation (details in: **[6]**, **[16]**)

$$W = W_V + W_H \approx 2\rho Vn \approx 2\rho\sigma\lambda n \approx 2(\omega^2 h / 8\pi^2 c\sigma n)\sigma\lambda n = h\nu \qquad (2.3)$$

Where: $V=\sigma\lambda$ is the volume of one "ring". The symbol ($\approx$) replaced with (=) due approximation in eq. (2.1) is conditioned with σ, n **[6]** that become cut. Thus, the initial key assumption/postulate of QM, ***eq.*** $W=h\nu$ ***gets a causal explanation as energy of quantum wave packet, derived from classical wave' peculiarities.*** These results with (*2.1.1*) show artificial separation of QM from classical/causal principles that are seen repeatedly.

3. Representation $W=mc^2$ *as a wave-vortex' energy*

We will show first that the seeming difference between classical expressions of energy/impulse ($W=mV^2/2$, $P=mV$) and equations ($W=mc^2$, $P=mc$) gets simple explanation with the "ring-strings" distribution of field's energy, peculiar to elementary particles. Let us remind first that equality of kinetic and potential energies $W_k= mv^2/2=W_p$ derives from balance centrifugal and potential forces for the movement by circle, in the central potential field. The total energy of balanced system becomes $W=W_k+W_p=mV^2$ that corresponds by form to $W=mc^2$. Thus, In view of form we can conclude that ***eq.*** $W=mc^2$ ***it evidences the vortex-dynamic nature of the basic substance.*** Presented interpretation well corresponds to the fact of existence of particles' spin $S=(\hbar,\ \hbar/2)$ **[6]**. We will show the wave-field origin of mass with the Bohr's hydrogen atom model, as the simplest way. The balance condition of electron on the first orbit with radius a_0 is the equality of centrifugal and Coulomb's forces:

$$m_e V^2 / a_0 = e^2 / 4\pi\varepsilon_0 a_0^{\ 2} \qquad (2.4)$$

We write from (2.4):

$$m_e V^2 / 2 = e^2 / 8\pi\varepsilon_0 a_0 = W_E \qquad (2.5)$$

W_E is potential energy of a system. Eq. (2.5) confirms the equality of kinetic and potential energies. We present as $a_0=\lambda_e/2\pi\alpha$, orbital speed as $V=\alpha c$, elementary charge as $e=(2\varepsilon_0\alpha hc)^{0.5}$ where λ_e is the Compton wavelength of the electron, ε_0 electrical constant. Substituting the values into (2.5), we get[5]: $m_e c^2 = hc/\lambda_e = h\nu_e = W$ (2.6)

Eq. (2.6) shows equivalence of electron's rest mass to quant of energy with Compton wavelength. It shows a wave-field nature of the particle.

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[5] We have accepted non-applicability of SR and Lorentz's coordinates' transformation at the subject; shown in several works (as in [8]).

4. Causal interpretation of de Broglie's wave: Physical meaning of "Feynman's arrows": Explanatory to particles' double slit interference

a). The electron's model and de Broglie wave

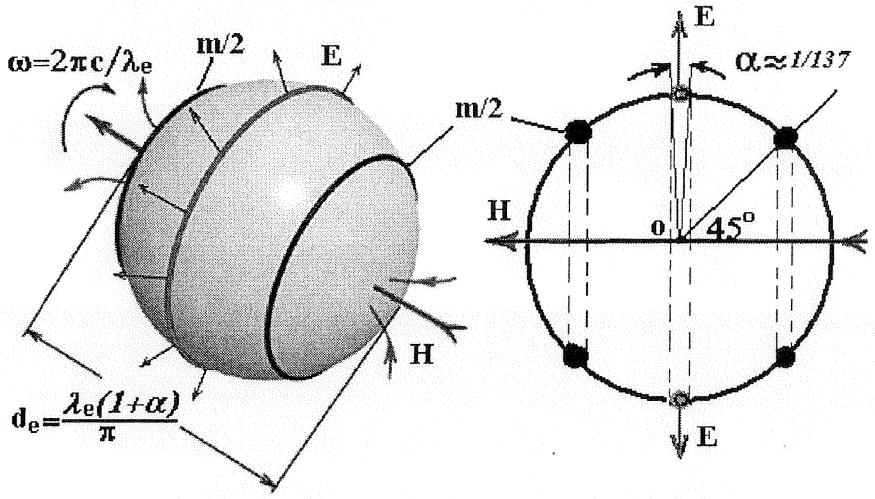

Figure 2. Schematic Model of the Electron

(*The color figure is available from site:* http://vixra.org/pdf/1208.0213v1.pdf)

Explanatory to Figure 2

Known peculiarities and parameters of electron deduced by modeling it as Compton's standing wave-vortex/interference **[6]**. $m_e=W_0/c^2=h/c\lambda_e$, $W_0=hc/\lambda_e$, $W_E=W_\mu=0.5aW_0$, $s=h/4\pi=\hbar/2$, $\mu_e/\mu_B \approx 1{,}00115965$, Where: W_0 energy of main pick of interference that manifests as rest mass of the particle, W_E, W_μ are energies of electrical and magnetic pseudo static fields conditioned with secondary picks of interference, a fine structure constant. μ_e, μ_B electron's mag. moment and Bohr magneton.

The electron's model is presented in **[6]** as circularly polarized Compton's standing wave-vortex/interference (Fig. 2). Its experimentally known peculiarities and parameters are interpreted there within modeling (chap. 3), **[6]**. De Broglie's wave is interpreted as Doppler Effect ("wave' beating") conditioned with movement of localized particle $\lambda_D=\lambda_c(c/V)$. We present additional clarifications to de Broglie wave essence using graphic images (Figure 3). The origin of "ring-strings" of de Broglie wave becomes simply explainable from Maxwell's equations as field's induction, in consequence of movement of the Compton's standing wave-vortex. The amplitude of tension of induced field (in the "ring-strings" of de Broglie wave) is simply defined as $E_D = E_0 dx/cdt = E_0(V/c)$ where E_0 is the amplitude of tension in Compton wave circulation (2.1).

We define the energy for de Broglie's wave packet corresponding to whole wavelength λ_D [Fig. 3. *a)*], **[6]** using the same reasoning (*2.1.2.*)

$$W_D = 0.5h\nu_c(V^2/c^2) \qquad (2.7)$$

The tension of field increases with the origin of new standing "rings" from moving particle at the interval *1-2.* It reaches to a maximum at the point *2*. The tension decreases with further movement of the particle, in consequence of new rings neutralizing the previous ones because of inverse directions of circulations (at the interval *2-3*). The field's tension becomes *0* at the point *3*. Described process quantitatively corresponds to radiation of photon and its further back-absorption by freely moving particle in the alternating intervals accordingly. The acting/average value of de Broglie's wave energy becomes half of its maximum that explains the coefficient *0.5* in eq. (2.7). Presented model causally interprets how moving particle "interacts with himself" [Fig. 3. ***c), d)***] that looks as "mystery" in QED (see ***exchange character of interaction, Feynman's arrows***). The presented interpretation clarifies productivity of QED as conditioned with its correct cause-effect base (which remains unformulated because of formal methodology!)

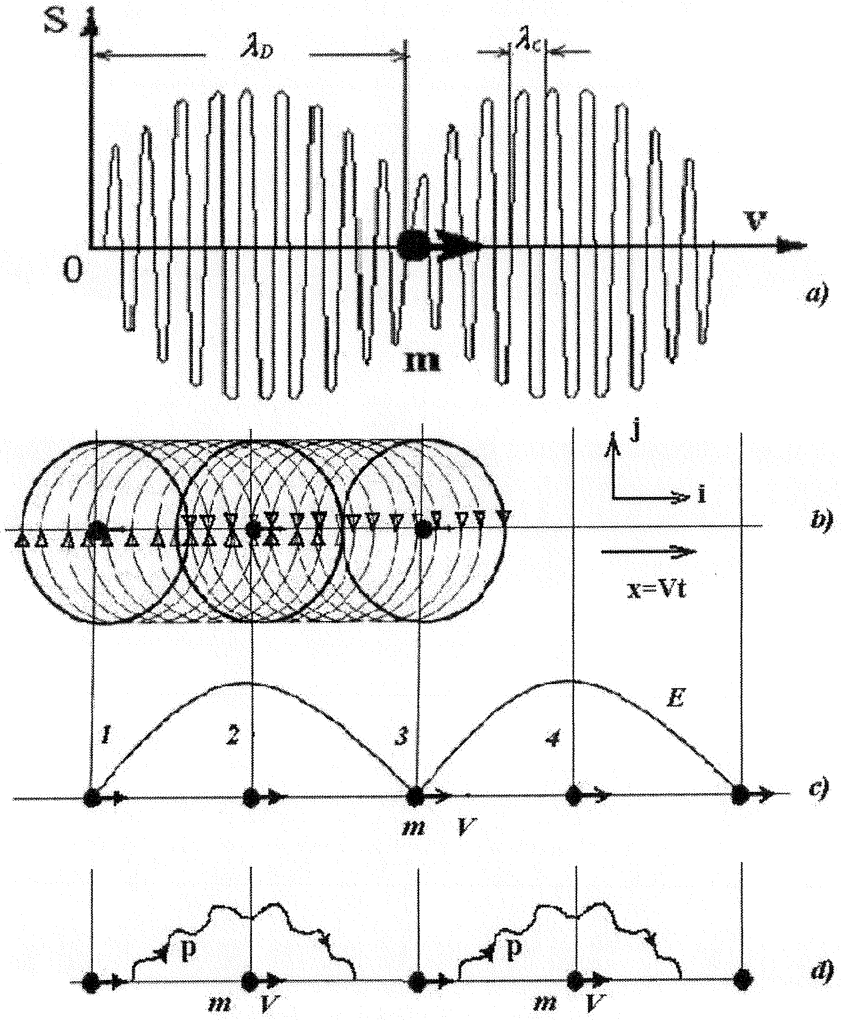

Fig. 3. De Broglie's wave illustration

Explanatory to figure 3

a) The graphic of "wave beating". ***b)*** The origin of standing "ring-strings" of de Broglie's wave from moving particle at the interval ***1-2***, and their mutual annihilation at the interval ***2-3*** (the annihilation' process took place in range of middle solid circle where direction of circulations become contrary). ***c)*** The acting value of field' tension of "wave beating". ***d)*** Representation of a process as the alternate radiation/absorption of virtual photon ***P*** at the way of movement; it clarifies the physical meaning of ***Feynman's arrows*** in QED. (The "Ring-strings", perpendicular to drawing plane, not shown)

b). Particles' double slight interference

Presented interpretation of de Broglie wave gives opportunity of causal explanation of particles' double slight interference as well, which was one of the intriguing problems prior to QM. As we mark, De Broglie-Bohm theory quantitatively describes the phenomenon (*1.1.1.*) We present descriptive-imaginary representation of that within (Fig. 4.) Let us remember first that the main question, exciting the minds, was "how the single particle passes troughs two slights at the same time?"

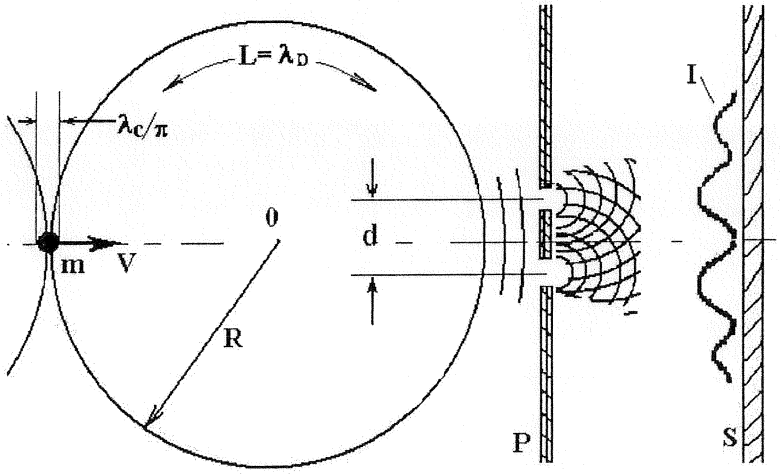

Fig. 4. Illustration to double-slit interference of particle

Explanatory to Figure 4

P partition with two slights, ***S*** screen, ***I*** a curve of interferential distribution on the screen, $L=\lambda_D$ one of "ring-strings" of de Broglie's wave, ***m, V*** mass and velocity of particle, λ_c Compton's wavelength of the particle. ***d*** distance between slights.

The front of de Broglie's wave reaches to slights before of particle. It causes the interference behind the partition and the distribution field on the screen. Considering $V<<c$, $\lambda_D>>\lambda_c$ and $R>>d$ we can accept wave front as "parallel" to partition plane. I.e. the deviation of particle's trace in range of ***d*** will not influence the interferential picture. I.e., single particle can pass by any of the slights that doesn't change distribution picture on the screen. Interferential picture becomes disturbed with closing any of slights.

The problem looks as follows; if we assume that the single particle passes through one slit only the difficult question arises then: "why interferential picture from single particle is disturbed when we close the "unnecessary" second slit?" We get the simplest answer if we ignore DP instruction and look at particle together with de Broglie wave (as it is actually!) Having in view the "interference of de Broglie wave" instead of "interference of the

particles", we get easy explanation of the problem. Thus, we should just take care, that there are two participants in phenomenon and not only one. We can find clear causal interpretation of this intriguing issue as;

de Broglie wave causes the interferential distribution that directs the particles within configuration of standing waves field

With the same, one of the main difficulties, that force the introduction of "new laws of nature" (probable/statistically), goes out from the agenda.

5. The causal interpretation of wave function and SE

a). Let's assume that electron moves by circle equal by length to de Broglie's wavelength. For the electron $\nu_c=\nu_e$, on a first Bohr's orbit $V/c=\alpha$ we get from eq. (2.7):

$$W_D = 0.5a^2 h\nu_e = 0.5a^2 hc/\lambda_e = h\nu_R = hR_y \qquad (2.8)$$

Where $\nu_R \approx 3.28\cdot 10^{15} s^{-1} \approx R_y$ is the Rydberg's constant. We represent a density of energy for de Broglie's wave accordingly to (*2.1.2.*), (*2.1.4.*) as:

$$\rho = \langle \mathbf{E}_D \rangle^2 = \langle \mathbf{E}_0 \frac{V}{c} \rangle^2 \approx \frac{h\omega_e{}^2 V^2}{8\pi^2 c^3 \sigma n} \cdot \langle \mathbf{i} \sin\omega_e t + \mathbf{k}\cos\omega_e t \rangle^2 \qquad (2.9)$$

From eq. (2.8), we write:

$$W_D \approx \int_v \rho d\upsilon = \int_v \langle \mathbf{E_D} \rangle^2 d\upsilon = hR_y \qquad (2.10)$$

We replace the vector' square in eq. (2.9) as:

$$\langle \mathbf{i}\sin\omega_e t + \mathbf{k}\cos\omega_e t \rangle^2 = \exp(-i\varphi)\exp(i\varphi) \qquad (2.11)$$

Using eq. (2.8), (2.10), we can represent W_D in form:

$$W_D \approx \int_v \Psi\Psi^* d\upsilon = \int_v \langle \mathbf{E_D} \rangle^2 d\upsilon = hR_y \qquad (2.12)$$

Where $\Psi = A\exp(-i\varphi) + B\exp(i\varphi)$ is a complex function required to find.

Comparing (2.11), (2.12) with (1.1), (1.4), using $(-i\varphi) = (-\frac{i}{\hbar}\sqrt{2m_e W}\, x)$, $W=m_eV^2/2$, $x=Vt$, $m_e = h\nu_e/c^2 = \hbar\omega_e/c^2$ (2.6) we get $\varphi = \omega_e t(V/c)^2$. We define required complex function as:

$$\Psi = E_0(V/c)\exp[-i\omega_e t(V/c)^2] = E_D \exp[-i\omega_e t(V/c)] \qquad (2.13)^6$$

Accepting the energy of de Broglie's wave (2.12) as measureless unit $hR\equiv 1$, using eq. (2.13) we can write:

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[6] We accept $B=0$ as it has no further application in the subject

$$\frac{W_D}{hR} \approx \frac{1}{hR}\int_v \Psi\Psi^* d\upsilon = \frac{1}{hR}\int_v \langle \mathbf{E_D} \rangle^2 d\upsilon = \int_v |A^2| dv \equiv \int_v |\psi^2| dv = 1 \quad (2.14)$$

Where: $|A^2| = \langle \mathbf{E_D} \rangle^2 / hR_y$. Thus, we come to eq. (1.4)

This representation clarifies the meaning of eq. (1.1), (1.4) and SE. In QM $|A|^2$ considered as "a probability to find the particle in elementary volume" (*1.1.2.*) and the "normalization" is applied there that expressed in eq. (1.4). We can see from (2.14) that named operation actually reduces to acceptance of $W_D = hR \equiv 1$. I.e., ***"normalization" actually means consideration*** (declaration) ***of the energy of de Broglie's wave as the "unit of probability"*** (or, "a full probability") ***for particle's location***. From eq. (2.13), we can see that ***the amplitude of wave function*** ("amplitude of probability") ***actually corresponds to the module of field's tension in de Broglie's wave***. It means; the measureless value of "probability of the particle's location" coincides with the density of de Broglie's wave energy in relative units. The distribution of de Broglie's wave energy has a form of "ring-string" [*2.1.4.a*).], [*2.1.5.a*).] It coincides with the orbit of particle's movement. Thus, the presented interpretation clarifies the compliance of QM results with the reality and its productivity. ***De Broglie wave and the particle are indivisible; hence, the distribution of de Broglie wave energy defines the location of the particle simultaneously***

b). Coming to the question of ***unjustified entrance of conjugate function*** ψ^* ***in eq.*** (1.2), we can see that mentioned operation actually serves to "organize" the desirable result (1.4). Meanwhile, similar solution (2.10) in real values derives naturally, based on QWE as definition of energy's density (without unexplainable suppositions and arbitrary operations!) Thus, we can assert; ***Schrödinger's equation derives from Maxwell's quantized wave equations***. We would mark that QWE actually contains fine structure constant and HU as well, since the parameters σ, $\Delta\lambda$, n are defined with α (*2.1.2.*) Moreover, α also derives from wave peculiarities; it is represented as the ***wave constant*** in **[6]**. We shall refer to **[10]** where the connection of α with HU and to wave interference is shown that independently confirms presented approach. The common roots of SE and ***Dirac'***, ***Pauli'***, ***Klein-Gordon' equations*** and ***Heisenberg's matrix mechanic*** are known from numerous works as **[8]**.

c). We have seen above that ***de Broglie wave is a secondary phenomenon rising as consequence of particle's movement*** (we image it "as a tail of particle" for logical comfort, if it is permissible!) The

cognitive mistake of intention "to build a localized particle from de Broglie wave" becomes obvious. The impossibility of solution has become clear for theorists by formal-quantitative reasoning. By presenting the standing wave-packet (particle) as series of Furrier' harmonics $U=\Sigma f_n(\lambda_{Dn})$ depending on velocity $\lambda_D=F(V)$, it becomes clear: ***the construction demands existence of numerous components, moving with different speeds, in the same place!*** Thus, impossibility of intention is obvious (the particle immediately decays). However, the wave-field concept of localized particle has been rejected totally, which has been crucial in physics! (*1.1.1*)

We get solution of the problem if we "build" the localized particle from Compton standing wave. Considering that ***Compton wave and its harmonics do not depend on velocity***, considering as well ***the equivalence of Compton wave energy with eigenenergy of particle*** ($m_oc^2=h\nu_c$) we get direct clue to the essence of localized particle **[6]**.

d). Coming to the issue of SE ***significance***, we would emphasize first that QM expresses the "external behavior" of the particle and does not reflect its full essence. Why is it so? SE actually describes the secondary phenomenon (de Broglie wave) arising from particle's movement (i.e. the behavior of "tail" only!), and the "owner" is actually not presented there. For example, considering $V=0$ for freely moving particle, we get $\lambda_D=\infty$; by the same we lose any date concerning to particle, i.e. our particle just disappears! However, within realistic description ***we should have the particle in rest condition***. It shows that SE ***is a part of some realistic description*** (as it expresses a part of real phenomenon only!) We shall present a complete equation, satisfying the mentioned requirement, for energy of the electron in first Bohr orbit in hydrogen atom, including fine structure level (i.e. for $n=1$, $l=m=0$, $m_s=\pm 1/2$). We shall represent full energy of electron as: $W = w_R + w_o$ where: w_R, w_o are the eigenenergy and orbital energy, accordingly. We shall present energies of mass, electrical and mag. static fields of electron, using transformations (2.5), (2.6), considering equality of electrical and mag. static fields' eigenenergy **[6]**. Expressing these by Compton wave frequency ν_e and α, we write full eigenenergy as:

$$w_R = w_m + w_e + w_\mu = h\nu_e + 0.5\alpha h\nu_e + 0.5\alpha h\nu_e = h\nu_e(1+\alpha) \qquad (2.15)$$

Where: w_m, w_e, w_μ are energies of mass, el. static ("charge") and mag. static fields accordingly. We shall present orbital energy using (2.5), (2.8), expressing fine structure energy w_f in same units:

$$w_o \cong w_k \pm w_f = \alpha^2 h\nu_e/2 \pm \alpha^4 h\nu_e/8 \qquad (2.16)$$

The sign (±) for fine structure energy considers spin-orbital orientations (m_s). We get below expression of full energy:

$$W \cong h\nu_e[1+\alpha+(\alpha^2/2)\pm\alpha^4/8] \qquad (2.17)$$

A *3*-rd member in eq. (2.17) is the kinetic energy $w_k=m_eV^2/2=\alpha^2h\nu_e/2=hR_y$ that is equal to de Broglie wave energy (or, Rydberg energy). It defines energetic terms of electron. Using *n*, $m_s=\pm1/2$, we can write (2.17) as:

$$W \cong h\nu_e[1+\alpha+\alpha^2/2n^2\pm(\alpha^4/2n^2)(3/4n-1/2n)] \qquad (2.18)$$

The eq. (2.18) differs from accepted standard description of spectral terms by presence of first two members. These present the mass' and static fields' energy of the electron. Next members depend on speed. Considering $V=0$ these become *0* and eq. (2.18) turns to eq. (2.15) presenting eigenenergy of electron. Thus, eq. (2.18) reflects the complete phenomenon as it takes place in reality. The components of orbital energy (without additional small corrections) we can present as:

$$W_k = h\nu_e(\alpha^2/2n^2) = (hc/\lambda_e)(V/c)^2(1/2n^2) = hR_y/n^2 \qquad (2.19)$$

$$W_f = \pm(hR_y/n^2)(\alpha^2/4n) \qquad (2.20)$$

These correspond to QM results as shown above. The spectral energetic terms can be presented as:

$$W_o \cong h\nu_e(\alpha^2/2n^2)(1\pm\alpha^2/4n) \qquad (2.21)$$

Analogical representations are possible for orbital radiuses and speeds:

$$R_n \cong R_0n^2(1\pm\alpha^2/4n),\ V_n \cong V_0/n, \text{ where } R_0 \cong \lambda_e/2\pi\alpha,\ V_0 \cong \alpha c \qquad (2.22)$$

Equations (2.18), (2.21), (2.22) present the energetic, geometrical and dynamical peculiarities (behavior) of the orbital electron as functions of its single eigen parameter (Compton wave). The opportunity of adequate description of electron's behavior on causal basis is shown above content. Thus, the quantitative consequences of QM and causal representations coincide if we replace the relative values ("probable-average") with the real physical. The realistic form and "workability" of eq. (2.18) clarifies significance of SE. We can define the cognitive essence of SE as, ***Schrödinger's eq. is a partial description of real phenomena in relative units*** (by the same, it leaves place to arbitrary interpretations!)

6. The possibility of minimizing used concepts/system factors

a). We start with brief retreat. In accepted courses and academic publications we often come across some ordinary sub explanatory as, "here *c* is a speed of light", and "*h* constant of Plank" (!) Such presentation reflects cognitive confusion in modern methodology as the subject of

study initially becomes unclear (with what we deal?) It changes nothing if we use other accepted expression: "*h* is the World Universal Constant" as the issue remains open (a constant of what?) The matter is the problem has a cognitive character, which requires logical solution and it is banned by accepted rules! The initial approach to the problem is presented in [*1.3.1. b)*], (*1.4.3.*) Our next example relates to fundamental concept of "elementary charge". It is possible to comprehend from logical viewpoint that this concept actually reduces "the drop of weightless el liquid" that never has existed! We "saw" in fact, the particle as the source of electric (magnetic) static fields only, and nothing else! We never "saw" the independently existing kinds of realities as "pure" el. (or, mag.) static fields, separate from mass, spin and other properties of particle. Hence, we should accept the static fields as the indivisible peculiarity of particle and do not to input other hypothetical things ("charge" that have the "especial duty" only of creating static fields!) I.e., the "elementary charge" actually belongs to large family of hypothetic realities such as the unobservable "ethers", "kinds of liquids", "phlogiston" etc that have brought confusions and have only aggravated the problems. Thus, we can just remove the concept and term of "elementary charges" from our lexicon generally, if we wish to express the facts as they are! Opponents will point on the existence of constant $\pm e$ as "indisputable evidence" to reality of "elementary charge". Such argumentation is simply a consequence of initial confusion in methodology as an aftermath of mixing the real physical objects with mathematical (we can compose different kinds of quantitative invariants, with combination of corresponding variables and announce "new kinds of realities"!) Actually, we already have got rid of the concept of "el. charge" in above content as in eq. (2.21), (2.22). We shall bring one important remark on that. With application of "el. charge" as "source of static field", the necessity of applying "special settling coefficient" (system constant) arises there that gives opportunity to link the real-physical value with the hypothetical (i.e. a field, with "charge"). ***The electrical constant ε_0 serves for it, the existence of which remains completely unclear in contemporary physics!*** As we saw above, with representation of static fields' energy as part of localized QEF's energy, the concepts of "el. charge" as well as the system constant ε_0 automatically go out from our formulas **[6]**. Thus, we can comprehend that these concepts generally were unnecessary; that much simplifies our formulas and the picture of the subject area. We can comprehend that there is nothing strange from cognitive point. The necessity of changing used concepts and

their quantities arises with the passage from macro to microcosm (such as the concepts of "temperature", "color", "pressure", etc become unclaimed and others appear there). Thus, we should be ready to "sacrifice" our usual concepts under dictation of facts and logic. The same reasoning is applicable to "mag. charge" (or, "Dirac Monopole")[7]. As example, we shall present Coulomb force in rational form, for two elementary particles, acting on a distance $R>>\lambda_e$

$$F_C = \pm e^2 / 4\pi\varepsilon_0 R^2 = \pm\alpha\hbar c / R^2 \qquad (2.23)$$

Presented expression shows needlessness of concept "elementary charge" as well as the "electrical constant". We can judge with this example that kinds of similar "system constants" just serve to link false realities with the actual. Meantime, two basic natural constants h, c and geometrical/dynamical constants π, α (dimensionless) become enough for rational descriptions of phenomena in microcosm, in case of correct conceptual interpretation of reality, excluding kinds of "system constants". Presented formulas satisfy to this criterion (the same in **[6]**)

b). We can comprehend ***actual meaning of Plank's constant*** if we simply represent QEF as one single contour of field's circulation. We can judge from (*2.1.2*) and **[6]** that its energy's value and spin will be the same for the photon. For the electron (*2.1.1*), the distribution factors of energy and impulse moment arise (that cause a "charge", mag. moment and reduces spin on half), that we shall not examine, as these are not essential in our cognitive conclusions. In conformity to Maxwell's first law, we write:

$$u_\lambda = \oint_\lambda \mathbf{E_M}\, \mathrm{d}\mathbf{l} \qquad (2.26)$$

Where: u_λ, $\mathbf{E_M}$ the full potential of the contour and tension's amplitude vector accordingly, corresponding to above representation of QEF. To define quantized value of u_λ we see first that $(hc)^{0.5}$ has a dimension $[L^{1.5}M^{0.5}T^{-1}]$ that corresponds to measure of (*potential*) x (*length*). We accept:

$$\sqrt{hc} = const = u_h \qquad (2.27)$$

We call $u_h = \sqrt{hc} \approx 4.457 \cdot 10^{-13}$ $[V \cdot m]$ ***Quant of Potential*** (QP)

Using concept of QP we can represent QEF's energy as:

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[7] There is no argumentation, why "Dirac Monopole does not exist" yet and the "elementary charge exists", since these concepts are fully equivalent.

$$w = u_h^2 / \lambda = hc / \lambda = h\nu \qquad (2.28)$$

It follows from (2.26), (2.27)

$$u_\lambda \lambda = E_M \lambda^2 = u_h, \text{ and } E = u_h / \lambda^2 \propto \lambda^{-2} \qquad (2.29)$$

We get to most important conclusion, causally clarifying how the QEF "works". Eq. (2.29) shows: ***the wavelength and all parameters of*** QEF ***are defined with amplitude of wave tension.*** We shall emphasize the causal/classical character of established relation. Eq. (2.29) shows that QEF's oscillation frequency (wavelength) depends exclusively on field's tension. By the same, we can comprehend that "quantum laws" actually are manifestations of the cause-effect classical laws (that seems repeatedly). We shall mark next cognitively important conclusion from this representation: ***the linear dimension of elementary particle is inversely proportional to eigenenergy.*** Thus, we come to comprehension that the main principles of elementary particles' physical essence are:

The elementary particles are wave-vortexes of QEF. ***Their eigenenergies depend on field's tension: the higher the field's concentration, the smaller particles' dimensions and the bigger their eigenenergies.*** The photon corresponds to this principle on large experimental basis. The wavelength and interaction time (wave packet length) of photon decrease with rising of energy. By dominating opinions however, the localized light particles are considered to have "small sizes", compared to heavy ones, vs. to photon. We see such belief as intuitive-esthetical only, as there are no facts directly proving such imagination, while the arguments on inverse direction are much (if considered!) We will examine the issue further. We shall emphasize one remarkable detail as well. As we see, QP=$(hc)^{0.5}$ has "electrical dimension" (numerically equal to the potential of field's contour with unit length.) Such fact additionally confirms the wave-field nature of substance. Above representation shows: c ***and*** h ***are the invariant eigen parameters (dynamic and quantitative) that define peculiarities of*** QEF ***within combination of a single free parameter*** (as such the wave tension, wavelength, frequency etc. can be accepted). Using (2.29), we can simply estimate, for example, wave tension amplitude in QEF in its different manifestations. The field's tension amplitude in "ring-strings" of the electron (that forms a rest mass) will be about: $E_e \approx u_h / \lambda_e^2 \approx 7.5 \cdot 10^{10}\ [V/m]$. We estimate the field's tension amplitude for photons as $E_p \approx u_h / 2n\lambda^2$, where $n \approx (1/\eta\alpha)^3$ **[6]** number of pairs of whole wavelength in propagating wave packet. Accepting $\nu \approx R_y$, $\lambda \approx 10^{-6}\ m$ and $\eta \approx 1$, $n \approx 10^6$ (as atomic photons) we get:

$E_p \approx 10^{-7}[V/m]$. Thus, we can talk about a single kind of physical reality only forming the material world, which is the ***energy*** of QEF. We have seen above, other kinds of physical values are possible to represent as QEF's parameters, properties, peculiarities etc. (*1.4.3*), (*1.4.4*), and these is possible to measure with "electrical" units only. It means the possibility of removing the concepts of "mass", kinds of "charges" etc, as well as numerous "calibrating", "setting" factors and functions from our descriptions of microcosm, becomes real, which much simplifies the subject.

7. Causal representation of QEF's interactions

a). The contour imagination of QEF opens an opportunity to causal representation of its kinds interactions[8] (*1.3.5*). The interactions between QEFs with the same/close parameters (and their low harmonics) present actual interest to us, due to its wave essence (analogical to theoretical electrotechnique). We shall start with imaginary representation of atomic photons absorption/radiation that is important to comprehension of particles interaction in general. We shall mention first that QEFs' interactions will differ from classical: *1)* with restricted (quantized) character of actions, that turns the process to a transitional. The exact description of interaction becomes complicated due to appearance of non-periodical components and kinds of Fourier harmonics. *2)* The eigen parameters of QEF are simply interconnected by means of natural constants, which make results trivial, from cognitive viewpoint. We shall use mainly the second aspect; it yields approximate solutions/estimations in most cases. We present the description of photon's absorption/radiation process with hydrogen's atom within scheme (Figure 5.) We suppose the electron to be on the first Bohr orbit and frequency of incident photon equal to R_y to simplify the reasoning. The speed of electron $V_0=\alpha c$ and photon's frequency coincide with angular circulation in this case, i.e. $\omega_0/2\pi=R_y=\nu_p$ that presents classical resonance condition. The energy of photon transforms to electron with the parameter $\Delta w/w\approx 1/n$ per one cycle. The number of whole waves in the wave packet is $n\approx 1/\alpha^3$ ($\eta=1$), [(35), **[6]**] and we get $\Delta w/w\approx 1/n\approx\alpha^3$ that defines intensity of process[9]. We define the time of process (Fig. 5. *a), 1-2*)

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[8] That seems natural in context of unique base of substance.

[9] It is important to mark that α^3 plays in QED important role characterizing intensity of el.mag interchange.

as $\tau \approx 2nT = 2n/\nu_p \approx 2/R_y\alpha^3 \approx 1.6\cdot 10^{-9} s$. We define orbital parameters in the end of absorption considering orbital energy $w_e = w_0 + hR_y = 2w_0$ that yields $R_e = 0.5R_0$, $V_e = \sqrt{2}V_0 = \sqrt{2}\cdot\alpha c$ (the centrifugal and electrical forces balance preserves; their values increase four times). The single factor that qualitatively distinguishes excited condition of electron vs. former stable, it is the difference between de Broglie wave and orbital length.

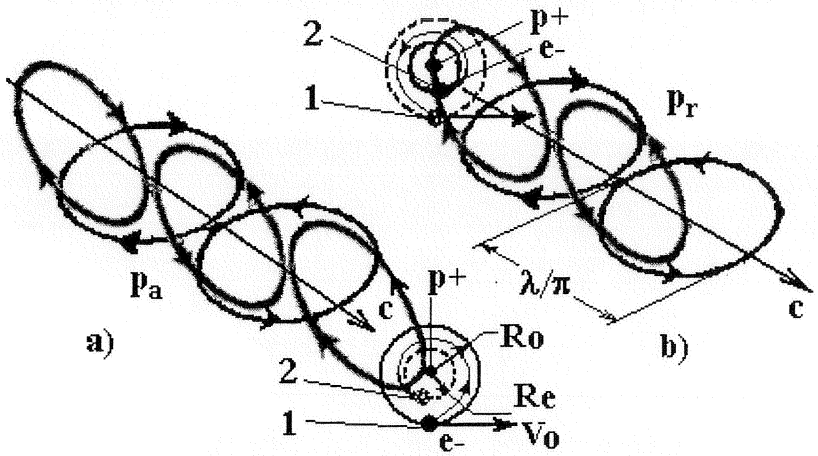

Figure 5. Illustration to absorbtion/radiation process of the photon

(*The color figure is available from site:* http://vixra.org/pdf/1208.0213v1.pdf)

Explanatory to Fig. 5

a) Scheme of photon's absorption process, and **b)** The radiation process, P_a the absorbing photon, P_r the radiating photon, R_0 electron's orbital radius before absorption process, R_e orbital radius in the end of absorption, p^+ proton, e^- the electron, λ wavelength of photon, V_0 orbital speed in unexcited condition.

The "ring-strings" of photon's wave packet reach to atom and pass through electron's orbit R_0. It induces additional component of tension in the contour of de Broglie standing wave (that coincides with the orbit) by the same frequency of electron's circulation. The increased tension of field accelerates the electron and "presses" it to center. The electron moves spirally and in the end of process reaches to radius $R_e < R_0$, its speed becomes $V>V_0$. The process corresponds to pumping energy and transition of electron to excited condition. **b)** Radiation of photon takes place with orbit's extension, and decreasing speed of the electron to the former.

Its relation becomes now $\lambda_{De}/2\pi R_e = \sqrt{2} > 1$. It means standing condition of wave becomes disturbed on the new orbit that "inhibits" movement of the electron. The radiation process starts, corresponding to movement of the electron with extending spiral trajectory (Fig. 5. *b), 2-1*). We shall

estimate tangential force and corresponding wave field's tension component, acting on the electron at the transition process as:

$$\langle F_a \rangle = dP/dt \approx m_e \frac{V_e - V_0}{\tau} \approx m_e \frac{ac(\sqrt{2}-1)}{1.6 \cdot 10^{-9} s} \approx 0.56 \cdot 10^{-15} N \qquad (2.30)$$

It corresponds to field's average tension $\langle E_a \rangle = \langle F_a \rangle / e \approx 3.5 \cdot 10^3 V/m$.

We can estimate length of spiral way as: $L \approx \tau(V_0 + V_e)/2 \approx 4 \cdot 10^{-3} m$.

We shall define the change of potential for de Broglie wave contour as: $\Delta u_D \approx \langle E_a \rangle L \approx 14\ V$. It corresponds to electron's energy change at *14 eV* that is close to R_y (*13.53 eV*). We shall link accelerating tension to absorbing/radiating photons wave tension E_p as:

$$\langle E_a \rangle \approx 2nE_p(\lambda_p / \lambda_D)^2 \qquad (2.31)$$

Eq. (2.31) gets easy explanation with imaginary representations of QEF and photon-electron interaction (Fig. 1, Fig. 5.) The wave packet' contours become "pressed and wrapped" (trapped) on the electron's orbit at the end of absorption. It means the superposition of *2n* numbers of contour λ_p on de Broglie wavelength λ_D takes place. Considering $\lambda_p/\lambda_D = 1/\alpha$ in first orbit and eq. (2.29) we get (2.31). Using values for E_p, n [*2.1.6. b)*], we get from (2.31) $\langle E_a \rangle \approx (2 \cdot 10^6 \cdot 10^{-7}/\alpha^2) V/m \approx 3.75 \cdot 10^3 V/m$ that confirms first estimation ($(3.5 \cdot 10^3 V/m)$ and presented causal model of interaction by the same. The mutual correspondences and straight links between orbit's radius, speed, orbital impulse, etc with the spectral energetic terms are obvious in presented model. As example, we can write ***Balmer-Rydberg formula*** as:

$$\nu_p = R_y(1/n^2 - 1/n_1^2) = R_y a_0(1/R_e - 1/R) \qquad (2.32)$$

Where a_0 first Bohr radius, R_e, R the radiuses in excited and stable orbits accordingly. Accepting $R_e = a_0$ (i.e. $L_{orb} = \lambda_D$) and $R = 4a_0$ ($L_{orb} = 2\lambda_D$), we get $\nu_p = R_y(3/4)$ that presents the first line in ***Lyman series*** [*2.1.5.d)*]. The same reasoning is possible for other frequencies/orbits as well. The opportunity of similar interpretations of phenomena conditioned with mutual influences of atomic electrons has to be clear for the reader. Descriptions of spatial orientations, precessions etc become technically difficult, due to number of participants. Different approximation methods are used in QM on the problem (see: ***quasi-classical approximation***, ***perturbation theory***), which principally are similar to causal-classical **[11].** We find interesting to mark the opportunity of pumping energy by low frequency photons to generate a high one (for example, the pumping the

environment with the red light to radiate green, used in some lasers), it becomes simply explainable within modeling. The contour's resonance to harmonics well confirms this case. In favor of presented model, we shall mark also that the retention ("trapping") of the electron in photon's interferential field has been studied theoretically and experimentally in numerous works **[12, 13].** This indirectly confirms the lawfulness of presented model in simplest logic. Since the photon's interferential field catches the electron, the opportunity of inverse phenomenon has to be, as any influence is mutual. The absorption of photon with atomic electron is possible to consider from this viewpoint. We shall mark also that ***photon's propagation process*** well fits to above interpretation. To preserve propagation speed equal to *c*, energy transfer in the photon's wave packet has to be with parameter $\Delta w / w \approx T / \tau \approx 1/n \approx (\eta\alpha)^3$ (35), **[6]** (i.e. with the energy, corresponding to single contour, per cycle). The process of propagation corresponds to induction of new contour "ahead" of wave packet with simultaneous annihilation of the last in the "end", per period of oscillation. The time of absorption/radiation process of photon corresponds to its propagation in presented model. The eq. (2.32) shows direct connection of spectral terms & orbits' radiuses. The speculative essence of "probable distribution" of atomic electrons becomes obvious by the same. As it seems, QM ***accepts the exact distribution of spectral lines on the screen, as visible*** (i.e. as fact) ***and rejects the same for the electrons in the atoms, since it is not subject to direct observation!***

The actually coincidences of "probable redistributions" (*1.1.5*) with experimental dates, concerning to single, double and *N*-slits interferences are shown in **[10]**. The essence of QM equations becomes evident once more by the same, as ***causally determined***, since the wave interference is a classical phenomenon, described within causal relations. The common principle of wave phenomena and electrons' behavior in atom becomes obvious, upon multiple judgments and results. The "probability" becomes then as an unjustified introduction, that causes confusion only. The contour representation of QEF appears productive to understand the interactions of localized particles also. The electron's model (Fig. 2) corresponds to contour representation. The representation of electron's interaction with γ quanta (***Compton Effect***) becomes simply possible by the same causal principle, as the photon with de Broglie standing wave.

Thus, ***the principal combination of energetic, kinematic and geometric properties of*** QEF ***in the common causal frame of electrodynamics is***

possible. The opportunity to reduce all kinds of phenomena to the electromagnetic becomes evident by the same (*1.3.5*)

8. Causality & uncertainty: Stability of elementary particles

The uncertainties of quantum objects are presented in official physics as "indisputable prerogative" of QM, which are "not subject to causal interpretations". The HU interpreted in **[6]** as deviation of QEF parameters in consequence to wave packet restriction. The direct communication of HU with the fine structure constant is shown. It has been shown in **[10]** also. However, we need to emphasize that ***the process of deviation of*** QEF ***parameters is not subject to direct experimental confirmation due to general quantum limitations*** (it is explainable by simplest logic; this circumstance, itself, leaves place to cognitive speculations to justify the formal methodology!) Mentioned view corresponds to conclusions of deserved theorists **[14]** (doomed to oblivion!) We shall divide first the QEF uncertainties by the obvious criterion. We call ***eigen uncertainties*** the consequences of above-mentioned wave beating process which is a causal/regular deviation of QEF parameters. We accept as ***external uncertainties*** those, which are conditioned by movement of QEF relative to observer. These are the uncertainties of coordinate-impulse and derived relations, connected to the process of measurement. We just note that their causal representations also become possible considering the wave nature of quantum objects, their actual sizes, time of measuring process, etc. We will study the first type only as necessary for future issues. The SUD (symmetry of uncertainties distribution) serves as stability condition of photon, expressed as $(\Delta\lambda/\lambda)^3 \approx (\eta\alpha)^3 \approx 1/n$ **[6]**. In geometrical-imaginary representation it corresponds to the symmetrically/homogenous volume distribution of wave-beating deviations on the full length of wave packet, with the parameter α (we find useful to image a chain having the same thickness and form of rings). It is taken in SUD $\eta = 1/2\pi$ (that yields $n \approx (2\pi/\alpha)^3$, $\nu \approx \nu_e$) as special (additional) condition necessary for stable localization of QEF. We imagine above condition as a possibility of homogenous chain to wrap on a wavelength with creation of a volume, symmetrical in respect to Cartesian coordinate axes. It corresponds to symmetric form, from geometrical viewpoint, and with well-balanced (in relation to 3-axes) rotating flywheel in mechanical analogy. Presented model of localized QEF corresponds also to ***phases and spatial angles equality condition*** of wave interference, from wave-dynamic point of view. It yields electron's parameters,

corresponding to actual (Fig, 2), **[6]**. From previous narration, we can formulate stability condition for kinds of QEF manifestation as below: ***a). The stability condition of none localized*** QEF (photon) ***corresponds to*** SUD ***by one dimension*** (the homogenous distribution of uncertainties by the length of wave packet). ***b). The*** SUD ***for the plan localized*** QEF ***is two-dimensional*** (for photon "trapped" in atomic orbit). ***c). The*** SUD ***for stable localized*** QEF ***is tree-dimensional*** (the localized stable elementary particles). The large spectra of freely propagating QEFs (photons, γ quanta), the limited quantity of atomic orbits and the existence of few stable localized QEFs only (known localized fundamental particles) become causally comprehensive in context of above presented common principle of SUD. We see one more time the whole importance of understanding the true meaning of α **[15, 16]**.

9. Wave-vortex representation of the basic hadrons & neutrino:
a). Physical/geometrical models of proton/neutron

The presented stability condition of electron $\eta_e = 1/2\pi$ corresponds to "wrap" of uncertainties by diametric length of localization $\pi d = \lambda$. We can understand it as a special case only, since SUD preserves also in cases $\pi D = k\lambda, \pi d = \lambda / m$, where: $k=1, 2, 3,...n$ whole positive numbers, and $m=2, 4, 6, ...(?)$ the even numbers (that can be a few only!) Mentioned conditions simply correspond to ***many-wave*** and ***half-wave*** (quarter etc) ***standing waves*** (interference). Possible kinds of QEF localizations are illustrated graphically. Electron corresponds to ***One-wave*** localization of QEF within illustration [Fig. 6. ***a)***] studied in **[6]**. We begin with study of the many-wave localization. As we see in graphic, the k number of whole waves becomes necessary to cover diametric length of localization πD in the many-wave localization; the cellular structure arises in result [Fig. 6. ***b)***.] We can represent each cell of localization in analogy to electron and describe as it is.

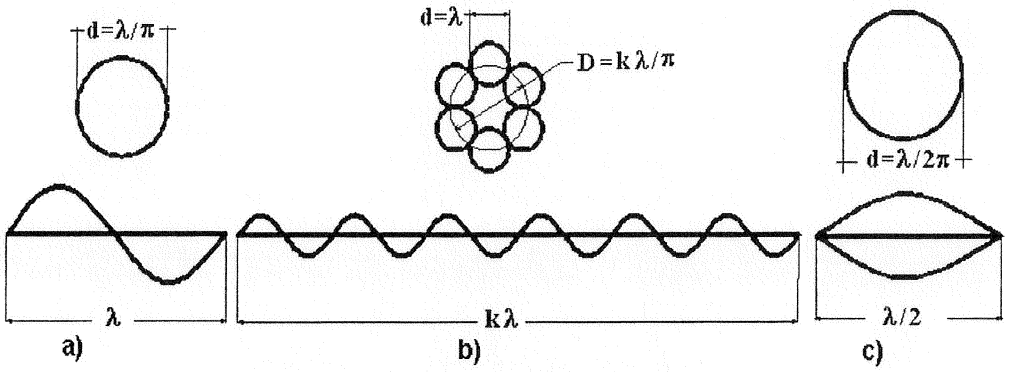

Figure 6. Possible kinds of QEF localisations

a) One-Wave, b) Many-Wave, c) Half-Wave standing waves

We shall emphasize first one important detail: as seen, the diameter of cells becomes $d_c=\lambda$ whereas for the electron it is $d_e=\lambda/\pi$. It yields $\eta_c=\eta_e/\pi=1/2\pi^2$. The number of whole waves in each cell will be defined as $n_c\approx(1/a\eta_c)^3=(2\pi^2/a)^3\approx\pi^3 n_e$ **[6]** to preserve SUD (see previous point) where: n_e is the number of whole waves composing electron's wave packet. Meantime, the reduction of wavelength by $1/\pi$ yields increase of energy on π^2 for the one single whole wave, in conformity to Maxwell's equations. Thus, the eigenenergy of the particle becomes $k\pi^5$ times more than electron's. It is easy to see that at $k=6$ we get known relation $6\pi^5 \cong 1836.118 \cong m_p / m_e$ close $(\delta \approx 0.016\%)$ to actual ratio of proton/electron mass **[17]**. The presented coincidence is not unique. Described model opens simplest opportunity to interpret large kinds of properties/peculiarities of basic hadrons (proton, neutron) that we shall examine with the scheme of 6-th cellular particle (Figure 7). We do not have an accomplished answer to intriguing question, why the number of cells is six? Below judgment deserves attention. The 6-cellular construction shows a specific (magical) peculiarity, from geometric viewpoint, which does not repeat with any other number of cells. An identical cell is possible to place inside the structure that touches all externals [Figure 7, ***a)***].

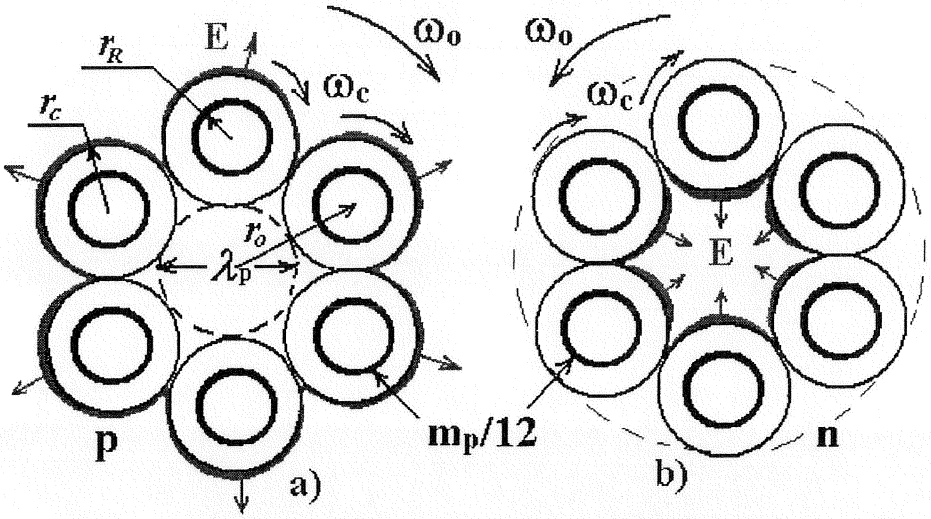

Figure 7. Scheme of the proton *a)*, neutron *b)*

m/12 - the "ring-string" of the mass, **E** – the el static field
ω_0 – orbital speed, ω_c – cellular circulation speed

(*The color figure is available from site:* http://vixra.org/pdf/1208.0213v1.pdf)

Mentioned peculiarity looks important from wave-dynamical point of view, as mandatory to stability of the many-wave standing condition.

Next conclusion concerns to dimensional appraisement of the particle. The particle "thickness" is equal to Compton wavelength: $\lambda_p \approx 1.3 \cdot 10^{-15} m$ as seen in graphics. It confirms experimental estimations of ***nuclear radiuses*** $[(1.3 \div 1.7) \cdot 10^{-15} m\,/per\ nuke]$[10]. Meantime, we get an important clue concerning to linking of hadrons in the nucleus structure and to its possible forms (to be discussed). The multi-cellular structure yields new peculiarities of particle. It may have the ***orbital momentum*** in two possible directions relative to circulation of cells, as seen in graphics. We can imagine two ball bearings as: ***a)*** rotating with the imaginary external corps, relative to unmoving central shaft, and ***b)*** rotating with the shaft, relative to unmoving external corps.

---

[10] The coincidence of nucleons "thickness" with Compton wavelength looks in modern physics as coincidental/unclaimed

We link the existence of basic hadrons *p, n* to the presented models; their different dates are interpreted as consequence of described two combinations of cellular/orbital momentums. We define summary spin of particles taking in consideration that angular speed of circulation in the cells decreases as the waves become "wrapped" by π times shorter vs. to electron scheme (Fig. 6). Considering above we get:

$\omega_c = c/\pi\, r_c$, $r_c = \lambda_p/2 = h/2m_p c$ where: ω_c , r_c are the angular speed and radius of cells accordingly. We define spin, using dates and graphic, as:

$$S = 12\frac{m_p}{12}\omega_c r_c \sin\frac{\pi}{4} r_c \sin\frac{\pi}{4} = \frac{h}{4\pi} = \frac{\hbar}{2} \qquad (2.33)$$

The particle's spin does not change, since the transformation of impulse/energy from "balls" to "corps"/"shaft" does not take place in both models. We present summary spin of particle as:

$$m_p(\omega_o r_o^2 + \omega_c r_R^2) = \hbar/2 \qquad (2.34)$$

Where: ω_o , r_o are the orbital angular speed and radius accordingly, and r_R is the radius of "ring-string" in the cells. In the first model: $\omega_o = \omega_c = \omega$ We find from graphic: $r_R = \lambda_p\sqrt{2}/4$, $r_o = \lambda_p$ and from (2.34) we get:

$S = 1.125\omega\lambda_p^2 m_p = \hbar/2$. Considering: $m_p = h/c\lambda_p$ we define angular speed:

$$\omega = c/4.5\pi\lambda_p \qquad (2.35)$$

The magnetic momentum of the particle is defined in analogy to electron **[6]** within below judgment: if we assume the particle does not have orbital momentum then its magnetic momentum has to be defined as: $\mu = (6/\pi)\mu_N$, since the contour of circulation of "el. charge" increases *6*-times and their angular speed becomes π times less, compared to single-cell particle (electron). ($\mu_N = e\hbar/2m_p$ is accepted as *nuclear unit magneton* in analogy to *Bohr magneton*). This simplest judgment yields the value definitely close to neutron's magnetic momentum:

$$\mu/\mu_N = (6/\pi)(e\hbar/2m_p)/\mu_N \approx 1.910 \approx \mu_n/\mu_N \approx 1.9130 \qquad (2.36)$$

Considering (2.35), we count the orbital factor of mag. momentum as:

$$\mu_o = eVr_o/2 = ec\lambda_p/2\cdot 4.5\pi = eh/9\pi m_p \approx 0.44\mu_N \qquad (2.37)$$

We have accepted r_o=λ_p in above judgment. By representing the "el. charge" (the pseudo static field) as manifestation of secondary interferential maximums **[6]**, considering also the ***phase/angular correspondence principle for the interference,*** we make following

conclusion: the secondary interferential maximums have to be located on the moving sides of contours of circulation. The instantaneous speed becomes maximal on the external side of cells, and zero on the internal in first model and inverse for the second. The "el. charge" has to be distributed on the external/internal sides of cells in the models accordingly ("charge" shown in red color). The factor of "charge" redistribution appears equal to the orbital. Meantime, for two models it gets inverse signs. We define the mag. momentum of two particles considering above judgment and results (2.36), (2.37) as:

$$\mu_p \approx \mu_N(1.91+0.44+0.44) \approx 2.79\mu_N \tag{2.38}$$

$$\mu_n \approx \mu_N(-1.91+0.44-0.44) \approx -1.91\mu_N \tag{2.39}$$

Where: the signs of mag. Momentums are taken relative to spin[11] **[18]**
The presented modeling demands ***presence of "charge", inside neutron, which is a new property*** that may be subject of an experiment.

b). Mass differences of *n, p* ***hadrons.*** The *p, n* masses' difference is linked to presence/absence of "el. charges" of particles within accepted interpretations **[18]**. Considering the previous content, we need to add only some details. Proton looks like "charged sphere" and neutron as "charged capacitor" in the models (Fig. 7). Hence, the difference of "electrical mass" will be defined with hidden electrical energy in the neutron (as a "capacitor"). The particles have difference in mag. fields' energy, due to difference of mag. momentums, which we also must take into consideration. We have defined Δm_{np}, considering models and above remarks by below formula:

$$\Delta m_{np} \approx \frac{e^2/8\pi\varepsilon_0 0.6\lambda_p}{c^2}(1+\frac{\mu_p-\mu_n}{\mu_n}) \approx \frac{\alpha h}{4\pi c 0.6\lambda_p}(1+\frac{\mu_p-\mu_n}{\mu_n}) \approx 2.3\cdot 10^{-30} kg \tag{2.40}$$

Where, we have accepted approximate radius of "charge" distribution inside neutron $\approx 0.6\lambda_p$ from graphics [Fig. 7. ***b)***.] We have used also the equality principle of electrical and magnetic components of pseudo static fields' energy ("charges") **[6]** and the known relation $e^2=2\varepsilon_0 ahc$.

---

[11] The ***isospin*** of inverse directions attributed to hadrons in particle physics, with the corresponding quantum number.

c). Differences of stability of** n, p **hadrons and possibility of their links

The dependence of magnetic force on distance is $1/R^3$ vs. electrical $1/R^2$. The magnetic-attractive and electric-repulsive forces between two electrons become equal on the distance of their centers $2R=\lambda/\pi$ (in touch state) with their axial orientation (Fig. 2). It shows that two electrons cannot be in linked state without external pressure. The Coulomb repulsive force exceeds the attractive magnetic if distance of two protons is bigger $l \geq (1.5 \div 2)\lambda_p$. In simple estimation, electrical force becomes smaller than magnetic, in the fusion of two protons (with "flat" sides). Their binding energy becomes comparable to the characteristic binding energy of nucleons[12]. Thus, the short-acting force between nucleons becomes possible to interpret as Ampere/Lorentz interaction of wave-circulation contours. It is possible to appraise from particles' schemes that the form of *n, p* hadrons and their mag. fields' configuration allow their different fusions. The preferable fusion of *n, p* hadrons, due to difference of "charges" distribution becomes comprehensive. We can conclude from models that stability condition of *n, p* cannot be the same. The electrical energy, concentrated inside, will be destructive for the neutron (in simple analogy to Coulomb repulsive force). The Proton's electrical energy is distributed in the space, starting from its surface hence is not destructive. Moreover, the Lorentz force, pressing the cells to center, increases stability of the proton. We estimate it as enough to keep the neighboring neutrons in stable condition as well, in the nuclear structure (Fig. 8). An intriguing question arises from presented nucleus model concerning to presence/absence of friction between linked particles. We suggest below interpretation, which simply follows from previous content and models. As it follows from HU representation, the dimension of particle also has the deviation (beating) as eigen uncertainties of parameters. It yields some enlargement in effective sizes of particle (it causes anomaly of mag. momentum of the electron) **[6]**. Meantime, it forms "boiling zone" on the surfaces and causes some bounce of particles within short distance, relative to their sizes. The particles become as free from friction due to described process. Thus, we can imagine the localized particles as "in vibrating and rotating state same time". We can see that combination of these properties with static Coulomb/Lorentz, central/axial forces opens an opportunity to causal comprehension of the formation of structure and behavior of atomic nucleuses. For example, it is easy to conclude that

---

[12] It varies in range about $1 \div 9$ MeV per particle, for different nukes

nucleons may have kinds of elastic deformations and, corresponding to these, the internal oscillations, axial and angular.

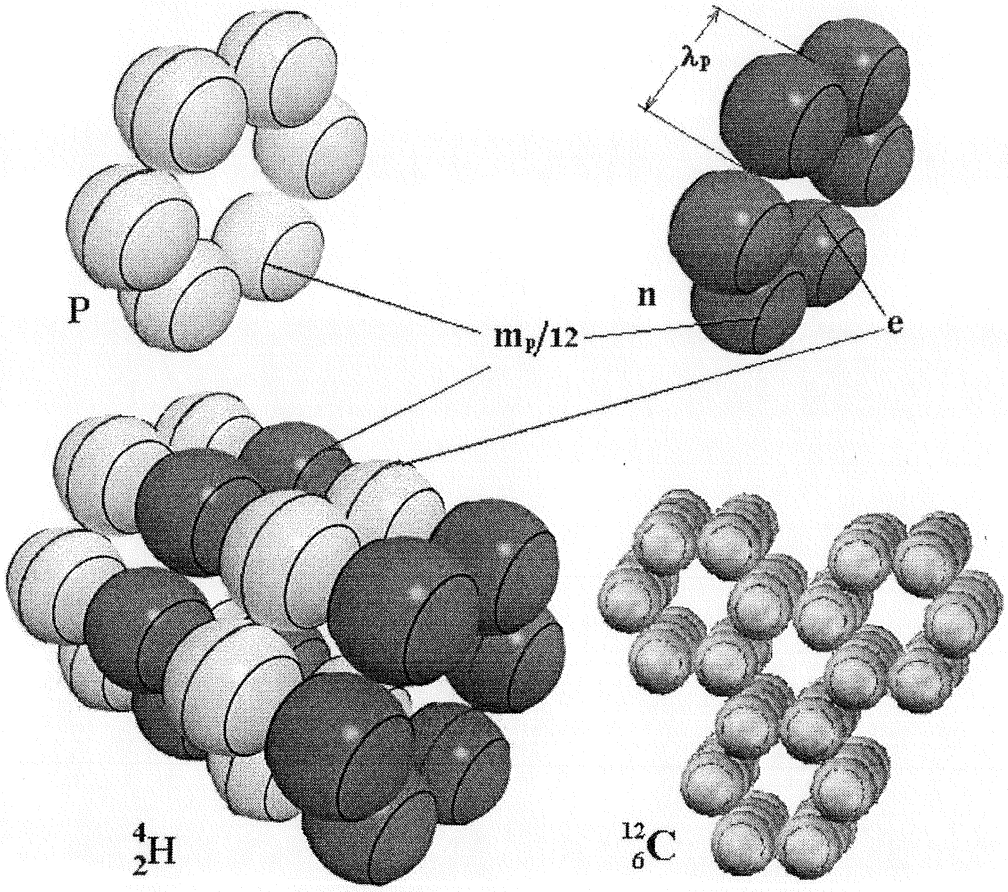

Figure 8
The schemas of proton, neutron, helium and carbon nucleuses

(*The color figure is available from site:* http://vixra.org/pdf/1208.0213v1.pdf)

The described nucleus model does not contradict to empirical models, used in practice of study of nuclear transformations ("drop" model, etc). The known common and individual properties of nucleus mostly correspond to the presented model that we shall mark briefly. As it seems, the *2p+2n* combination of hadrons has to be in high stability, due to symmetry of form and full balance of magnetic fields. It explains the high stability of ${}^{4}_{2}\mathrm{H}_{e}$, ${}^{12}_{6}\mathrm{C}$, ${}^{16}_{8}\mathrm{O}$ nucleons, etc (Figures 8, 9). A few strong

deviations of binding energy in the beginning of curve (light isotopes) and their gradual extinction on right become simply explainable.

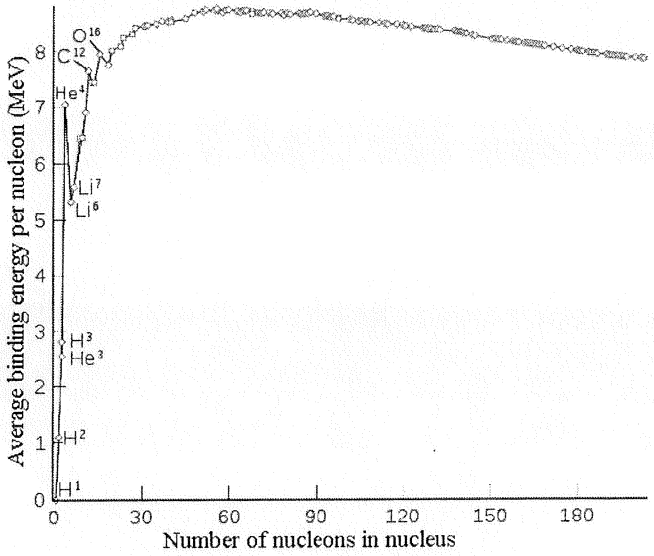

Figure 9. The curve of binding energy

The ***form factors*** and existence of ***magic numbers*** for nucleons, forming the stable isotopes **[18]** confirm modeling. As it seems, the nucleus model corresponds to ***hexagon-crystal structure***, formed by principle of fusion of the completed + uncompleted "tubes" of H^4 nucleons. The fusion of next nucleons may be from "side" (as in ${}^{12}_{6}C$) as well as from "front". The cells' mag. field plays "gluing" role in this case. We shall mark that presented ***nucleus model corresponds to quantum condensate*** as well, since "absence of friction" and presence of the static, polarized forces between nucleons. The model may help to explain the ***Bose-Einstein*** and ***Fermi-Dirac condensate*** properties as we hope.

d). Neutron's β decay. We can imagine the neutron as inverted (or excited) proton that aggravates its stability condition. It will be natural to look at the transition $n \rightarrow p + e + \tilde{\nu}$ as return of particle into stable condition from excited. We can imagine the process as reversal of the particle's orbital momentum into direction of cells circulation (Fig. 7). To describe it, we use analogy of photon's radiation. The angular speed of circulation and "charge" distribution (from internal to external) is changing in the process, similar to the first. The change causes induction

of new contours of circulation, forming two new localized wave-vortexes (particles) within conservation laws. The process quantitatively corresponds to photon's radiation/localization, i.e. it has a universal, ***exchange character.*** We use reasoning from energetic point of view (kinematical description of process looks somewhat difficult due to composite structure of particle). Assuming that process corresponds to photon's radiation, conditioned by α^3 factor [2.1.7. a).]), we define the energy transformation parameter as:

$$k \approx \alpha^3[1+(\mu_p-\mu_n)/\mu_n] \qquad (2.41)$$

In the formula we have considered a share of magnetic component (2.40) as well (vs. atomic photon's, where mentioned factor is ignored as insignificant, due *V<<c)*. We define the average power of energy change as: $d\varepsilon/dt \approx \Delta w/\tau \approx kw_n/T$, where $T=1\text{s}$ is unit time, τ relaxation time, w_n neutron's eigenenergy. Considering $\Delta w \approx w_e \approx m_e c^2$ (ignoring the share of neutrino in reaction), we get:

$$\tau \approx \frac{w_e}{w_n\alpha^3[1+(\mu_p-\mu_n)/\mu_n]}\cdot \text{s} \qquad (2.42)$$

It yields $\tau \approx 16\ min$ (vs. $\approx 15\ min$ experimentally established)
The correspondence of results (2.40), (2.42) to actual, deduced within general concept, evidences the reliability of interpretation and modeling.
We can imagine the process as "slow" creation of the standing wave-vortexes around the neutron, forming the electron and neutrino, in consequence of static fields' change, during orbit's reversal.

e). The essence of neutrino. We concluded from graphics [Fig. 6. ***c)***] that the 3-th type of localized QEF has a new kind of peculiarity vs. examined *e, n, p* particles. The secondary interferential peaks will be distributed with inverse-symmetrical angular directions, since the localization takes place as 1/2 (1/4, etc) wavelength of standing waves. It will cause mutual annihilation of secondary peaks and pseudo static fields in localization. It means, this kind of particles have no electric/magnetic "charge". We call these "truly neutral particles" vs. neutron which has a hidden electrical "charge" (and partially, magnetic). The stability of particles will be much easier vs. "charged", since these become free of corresponding interaction.
We will estimate eigenenergy of particles in general principle. We define the number of whole waves in a wave packet and energy using the same formula as for electron **[6]**, $n \approx (2\pi/ma)^3$ accepting $m=2,\ 4\ldots$ etc (for the

electron: $m=1,\ n_e \approx (2\pi/a)^3$). We define $n_1 \approx n_e/8$, $n_2 \approx n_e/64$ etc. These correspond to energies:

$$w_1 \approx w_e/8^2\ (\approx 8\ keV),\ w_2 \approx w_e/64^2\ (\approx 124\ eV),$$

Where: $w_e \approx 511\ keV$ is the electron's eigenenergy. The estimations of energy and absence of el. static fields ("charges") allow identifying described particles with the neutrinos. We cannot to assert it confidently due to absence of mag. momentum and the non-established values of masses (their presence?) for the neutrino. ***This kind of particles must have a mass,*** in general context of approach. It is interesting to mark that the radius of the first particle is about half of the first Bohr radius and second is about 30 times bigger of it. Thus, the neutrinos become incomparable with other localized particles with their huge sizes and insignificant densities of energy, relative to *n, p, e*. ***Mentioned circumstance and absence of "charge" allow neutrino to let other particles pass through it, "not feeling" their presence.*** Thus, the change of intuitive-ethical perception ("a light particle should be smaller than the heavy one!") allows simple solution of this old, intriguing problem - to explain huge penetration ability of neutrino.

f). Unstable particles & significance of their study

The presence of numerous kinds of localized unstable particles becomes explainable within previous content, as localized QEFs that not satisfy the SUD conditions [*(2.1.8. c)*.] The purposefulness of their further study also becomes easily understandable if guided by their common essence. The lifetimes of free *γ, e, p* basic particles, that practically may be considered as infinite even in cosmic scale (!), are incomparable with the unstable particles, which live for insignificant time, even in human scale (!) The natural question immediately rises for realistic researcher - why the lifetimes are so different for two objects, which are so similar by all other aspects? Mentioned huge difference and such inevitable question required an answer in same significance, before of detailed study numerous kinds of unstable particles. Their short lifetime and tremendous quantity may point on some construction defect or, unbalance etc vs. stable particles. ***Stable particles are incomparably important as objects of study vs. unstable, by the same criterion of their stability as completed constructions, which build our material world.*** Meantime, the possibility of mutual transformation of all particles, that yield few stable particles only in the end of any reaction, evidences the intermediate-transition

essence of unstable particles. Thus, the situation becomes analogical to below, from cognitive viewpoint. <<We initially know that water transforms into steam in result of boiling, and we do a detailed study of the process hoping to open water's chemical formula in such way (since we do not know other!) We become happy with every detection of new forms of bubbles or streams, imagining that we came close to the desired target, having no doubt that a new kind of concepts and science required to this purpose>>. The story gives an idea what is going on in advanced line of science and on what we can reckon in future on the subject, in author's view. Thus, we can assert. ***The numerous unstable particles do not deserve special attention due to their common physical essence, as kinds of transient states of the same substance.*** We can estimate the tremendous efforts for their production, systematization and description as a mostly useless occupation, upon above content.

3. Resume

- The opportunity of representation of the quantum phenomena in causality principle, on the wave-field common nature of the substance is shown, with numerous solutions of disputable problems.
- The interpretation of quantum objects' behavior as manifestation of "new kind" of nature laws ("probable/statistical") was an arbitrary-unscientific decision. The Schrödinger's equations and quantum laws are causal laws by their essence, related to electromagnetic field's description. These are derivatives from Maxwell's equations, with consideration of natural invariant parameters of field (c, h).
- Mentioned misinterpretation has brought to an artificial-speculative methodology (formal-mathematical) with rejection of natural way of research in subject area. The significance and productivity of returning to the causality principle and natural thinking is possible to judge by numerous results and self-consistency of the presented picture.
- The known viewpoint of Einstein on the essence of substance and his phenomenal prediction of the future way of physics, Schrödinger's and de Broglie's deep convictions in the wave-field nature of the particles, the efforts of many realistic thinkers on the same direction mostly become confirmed in presented work, in author's viewpoint.
- The cause-conceptual interpretation of substance may provide new aspects/opportunities to solution of basic problems, related to gravity, cosmology, symmetry breaking etc.

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SANTILLI'S NEW MATHEMATICS FOR CHEMISTS AND BIOLOGISTS. AN INTRODUCTORY ACCOUNT

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Abstract

Following decades of research, R. M. Santilli [1–5] has achieved a genotopic (that is, axiom inducing) lifting of 20^{th} century mathematics with a Lie algebra structure as used for reversible systems into a covering Lie-admissible form which is irreversible beginning with basic multiplications and related units, today is known as *Santilli genomathematics*. He also developed *isomathematics* through nonunitary, nonlocal and noncanonical transform suitable for dimensions 1fm or less level interactions. To deal the biological structures he formulated *hyper structured mathematics* that consists of multiple coordinate system for each axes. For example, the three dimensional Euclidean space is characterized by three coordinates one each for each axis. But now in hypermathematics we still have three dimensions but each axis is assigned n-number of coordinates. This helped, for example, in properly describing the growth of say seashells. To properly take care of antimatter systems at first quantization he further developed *isodual mathematics* of each one of the above new branches of mathematics. We have presented in the present review above stated branches of the new mathematics keeping the abstract contents at a bare minimum as title of the review goes because we are addressing

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this subject matter to the readers from chemistry and biology disciplines who wish to get themselves initiated into this new exciting scientific field. Moreover, the present author recently applied the genotopic branch of this new mathematics to nonequilibrium thermodynamics [6] and one of the outcomes of this exercise is the revelation of the existence of the four geno-thermodynamic times arrows.

Keywords: Hadronic mechanics, Iso-mathematics, Isodual mathematics, Lie-admissible mathematics, Hyper mathematics, Thermodynamics, Nonequilibrium thermodynamics.

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1 Introduction

All the time dependent processes are irreversible. This is so because the past times are unattainable. If a mathematical description of a time evolutionary process is *time reversal invariant* then it doesn't match the observations and hence implicitly and explicitly that mathematical description is inadequate to describe the said time evolutionary process. For example, Newton's equation (which is invariant to time reversal),

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F} \tag{1}$$

when describes the forward motion from say state A to state B the same equation does describe the motion from B to A following the same trajectory in the reverse order. But this then means that one would attain the past times. However, the forward motion is irreversible in time and hence the corresponding equation needs to be irreversible in time. This property is not inherited by eq.(1). Let us illustrate the time reversal symmetry operation that is effected by the substitution of $-t$ for t at a given position r in a dynamical equation. This has been described recently in adequate details [7] and its relevant part we reproduce below. Let the time reversal operation be denoted by $\stackrel{T}{\Longrightarrow}$ and operate it on the velocity vector u that illustrates the time reversal asymmetry of the velocity u as,

$$\frac{d\mathbf{r}}{dt} = \mathbf{u} \stackrel{T}{\Longrightarrow} \frac{d\mathbf{r}}{d(-t)} = -\frac{d\mathbf{r}}{dt} = -\mathbf{u}$$

that is u changes its sign hence it is not symmetric to time reversal. Thus the time reversal operation involves the change $(\mathbf{r},\ \mathbf{u},\ t) \Rightarrow (\mathbf{r},\ -\mathbf{u},\ -t)$. Now we see that eq.(1) is symmetric to time reversal that gets illustrated as,

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F} \stackrel{T}{\Longrightarrow} m\frac{d(-\mathbf{u})}{d(-t)} = \mathbf{F} \Longrightarrow m\frac{d\mathbf{u}}{dt} = \mathbf{F}$$

All dynamical equations that follow from eq.(1) inherit the same property and some of them are described below.

$$\begin{aligned}\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \rho \mathbf{F} \\ &\stackrel{T}{\Longrightarrow} \frac{\partial \rho(-\mathbf{u})}{\partial(-t)} = -\nabla \cdot (\rho(-\mathbf{u})(-\mathbf{u})) + \rho \mathbf{F} \\ &\Rightarrow \frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \rho \mathbf{F}\end{aligned}$$

where ρ is the mass density and now $\mathbf{F}$ is defined as the per unit mass conservative body force. Kinetic energy in Lagrangian form transforms as,

$$\begin{aligned}\rho \frac{d\frac{1}{2}\mathbf{u}^2}{dt} = \rho \mathbf{F} \cdot \mathbf{u} \quad &\stackrel{T}{\Longrightarrow} \rho \frac{d\frac{1}{2}\mathbf{u}^2}{d(-t)} = \rho \mathbf{F} \cdot (-\mathbf{u}) \\ &\Rightarrow \rho \frac{d\frac{1}{2}\mathbf{u}^2}{dt} = \rho \mathbf{F} \cdot \mathbf{u}\end{aligned}$$

The total energy, e (per unit mass), in Lagrangian form transforms as,

$$\begin{aligned}\rho \frac{de}{dt} = -\nabla \cdot \mathbf{J}_e \quad &\stackrel{T}{\Longrightarrow} \rho \frac{de}{d(-t)} = -\nabla \cdot (-\mathbf{J}_e) \\ &\Rightarrow \rho \frac{de}{dt} = -\nabla \cdot \mathbf{J}_e\end{aligned}$$

where $\mathbf{J}_e$ is the total energy flux density. Thus we see that all the above equations are time reversal symmetric.

However, the fluid dynamical internal energy, u, balance equation, namely:

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{q} - p\rho \frac{dv}{dt} + \mathbf{\Pi} : \nabla \mathbf{u}$$

poses serious difficulty in determining its time reversal symmetry because the terms $\nabla \cdot \mathbf{q}$ and $\mathbf{\Pi} : \nabla \mathbf{u}$ existing on its right hand side. On time reversal operation how these term would get transform remains ambiguous [7]. However, at the macroscopic level these terms are traditionally interpreted as the sources of irreversibility. In other words, the existence of fluxes $\mathbf{q}$ and $\mathbf{\Pi}$ and the gradient in $\mathbf{u}$ in the system means that the system is in a nonequilibrium state or evolving irreversibly in time. On the other hand,

in the absence of these terms the above internal energy balance equation is obtained as time reversal symmetric but still the evolution remains irreversible just because one cannot attain the past times.

Precisely the above fundamental difficulties led Professor R. M. Santilli to discover a proper mathematics. Following decades of systematic and sustained research Santilli [1] has achieved a *genotopic* (that is, axiom inducing) lifting of 20^{th} century mathematics with a Lie algebra structure as used for reversible systems into a *covering Lie-admissible form which is irreversible* beginning with basic multiplications and related units, today is known as *Santilli genomathematics*.

At the same time Santilli [1] has also achieved and identified an important subclass of genomathematics, also used for the description of thermodynamical processes of matter, which is an isotopic (that is, axiom-preserving) lifting of 20^{th} century Lie-mathematics with a Lie-isotopic structure, today known as *Santilli's isomathematics*. Finally, Santilli has discovered an anti-Hermitean map, called isoduality, for the description of antimatter, resulting in the novel *isodual genomathematics* and *isodual isomathematics*, that have permitted the first known initiation of thermodynamical studies for antimatter [8]. The study of biological structure evolution strongly suggested that one needs to describe them by using a higher dimensional mathematics although what one observes is the evolution in the three dimensional space that is perceived by the human senses. This led Santilli to construct *hypermathematics* and obviously the corresponding *isodual branch* (see for example the Refs.: [9, 10]).

Further, Santilli has proved the following basic theorem (see for example the Ref.: [5]), namely the **NO REDUCTION THEOREM:**

> *A classical system that is irreversible over time cannot be consistently reduced to a finite number of elementary particles all reversible over time and, vice-versa, a finite number of elementary particles all in reversible conditions cannot yield a macroscopic irreversible event under the correspondence or other principles.*

The above theorem has far reaching implications on bridging macroscopic and microscopic domains of physical descriptions. Santilli's new mathematics removed almost all inadequacies of all branches of physics, which were so far presented using 20^{th} century mathematics and mechanics. Hence, it is no exaggeration to state that now we have *New Sciences for A New Era*.

Moreover, as the subject matter of this paper is designed to serve the chemists and biologists no abstract details will be given in this paper. However, for all the intricate abstract and axiomatic mathematical details of these branches of the new mathematics the interested reader is advised to refer the references [1–5].

Thus the subject matter of this paper is divided into the following sections. In sections 2, 3, 4 and 5 we have given in an adequate details an elementary account of iso-, isodual-, geno- and hyper- mathematics. In this introductory account of Santilli's new mathematics we have added in section 7 (precisely subsection 7.5) a description of the recent proposal of the present author of genothermodynamic time's arrows (four in number) based on the second law of thermodynamics and its origin. Finally, the concluding remarks are given in section 8.

2 Santilli's Isomathematics

As the name goes *isomathematics* preserves all the axioms of 20^{th} century Lie-algebra but it introduces the non-unitary multiplication unit, which may be a scalar or tensorial quantity depending on the system in hand to be described. Of course, the possibility of using the unitary multiplying unit is of course allowed if a given system demands. In the following we proceed to describe the rudiments of *isomathematics* [1–5, 9, 11–17]

The lifting of the trivial unit $I = +1$ into a generalized unit, $\hat{I}$, is the first basic step of the isomathematics. Thus the lifting of I to N-dimensional *isounit* is represented as,

$$I = +1 \longrightarrow \hat{I}(t, r, \dot{r}, p, T, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \cdots\cdots\cdots) \tag{2}$$

where t is time, r is the position vector, p is the momentum vector, ψ is the wave function and $\psi^\dagger$ its transpose, and $\partial\psi$ and $\partial\psi^\dagger$ are the corresponding partial differentials. The positive definiteness of the *isounit*, $\hat{I}$, is prescribed by,

$$\hat{I}(t, r, \dot{r}, p, T, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \cdots\cdots\cdots) = \frac{1}{\widehat{T}} > 0 \tag{3}$$

where $\widehat{T}$ is called the *isotopic element* a positive definite quantity.

The *isonumbers*, $\hat{n}$, are generated as,

$$\hat{n} = n \times \hat{I}, \quad n = 0, 1, 2, 3, \cdots\cdots \tag{4}$$

Thus the *isonumbers* are $\hat{0}, \hat{1}, \hat{2}, \hat{3}, \cdots\cdots$. The above realization of *isonumbers* implies,

$$a^{\hat{0}} = \hat{I} = \frac{1}{\hat{\widehat{T}}} > 0 \tag{5}$$

$$\hat{a}^{\hat{0}} = \hat{I} = \frac{1}{\hat{\widehat{T}}} > 0 \tag{6}$$

Compare the preceding identities with that in the traditional algebra where we have $a^0 = 1$.

From eqs.(5) and (6) we see that there are two *isogroups*, in one of them the isonumbers are used and in the other the conventional numbers are used. The former is termed as the *isogroup of the first kind* and the latter is termed as the *isogroup of second kind* [18].

2.1 Some of the Basic Operations of Isomathematics

Notice that the use of *isounit* $\hat{I}$ constitutes the *nonunitary transform*. A simple way to preserve the said nonunitary transform is to express *isounit*, $\hat{I}$ as,

$$\begin{aligned} U \times U^{\dagger} &\neq I \\ \hat{I} = U \times U^{\dagger} &= \frac{1}{\hat{\widehat{T}}} > 0 \end{aligned} \tag{7}$$

and the Hermiticity of $\hat{I}$ is guaranteed by the prescription,

$$U \times U^{\dagger} = \left(U \times U^{\dagger}\right)^{\dagger} \tag{8}$$

where the superscript (†) denotes the transpose. Thus it implies that $U \times U^{\dagger}$ must be a positive definite N-dimensional matrix other than the conventional unit matrix and hence it is no where singular because we then have,

$$0 < \text{Det}\left(U \times U^{\dagger}\right) \neq I \tag{9}$$

Thus the lifting of unit matrix to isounit gets represented as,

$$I = +1 \longrightarrow U \times I \times U^{\dagger} = \hat{I} \tag{10}$$

which is the *noncanonical-nonunitary transform* of the conventional unit.

On the same lines the *isoproduct* is also a noncanonical-nonunitary transform of the conventional product, that gets illustrated as,

$$\begin{aligned} A \times B \;\longrightarrow\; & U \times (A \times B) \times U^{\dagger} \\ & = \left(U \times A \times U^{\dagger}\right) \times \left(U \times U^{\dagger}\right)^{-1} \times \left(U \times B \times U^{\dagger}\right) \\ & = \hat{A} \times \widehat{T} \times \hat{B} = \hat{A} \hat{\times} \hat{B} \end{aligned} \tag{11}$$

Thus we have obtained the following *isotopicliftings*, namely:

$$\times \;\longrightarrow\; \hat{\times} = \times \left(U \times U^{\dagger}\right)^{-1} \times = \times \widehat{T} \times \tag{12}$$

$$A \;\longrightarrow\; U \times A \times U^{\dagger} = \hat{A} \tag{13}$$

The *isoproducts* of isogroups of first and second kind get expressed as,

$$\hat{A} \hat{\times} \hat{B} \;=\; \hat{C} = A \times \hat{I} \times \widehat{T} \times B \times \hat{I} = (A \times B) \times \hat{I} = C \times \hat{I} = \widehat{AB} \tag{14}$$

$$A \hat{\times} B \;=\; A \times \hat{T} \times B \tag{15}$$

Thus $\hat{I}$ being a *multiplicative isounit* we identically have,

$$\hat{I} \hat{\times} \hat{A} \;=\; \hat{A} \hat{\times} \hat{I} = \hat{A} \tag{16}$$

$$\hat{I} \hat{\times} A \;=\; A \hat{\times} \hat{I} = A \tag{17}$$

that is the *isomultiplication* by the *isounit* doesn't change the iso- and conventional quantities. Compare this result with the conventional multiplication of a quantity by the unit matrix,

$$A \times I = I \times A = A$$

that the multiplication by the unit matrix doesn't change the quantity being multiplied. Hence, in general the *isomultiplication* by an *isonumber* doesn't change the iso- and conventional quantities that get depicted as,

$$\hat{n} \hat{\times} A = n \times \hat{I} \times \widehat{T} \times A = nA \quad \text{and} \quad \hat{n} \hat{\times} \hat{A} = n \times \hat{I} \times \widehat{T} \times \hat{A} = n\hat{A}$$

However, the *isoproducts* are not necessarily *isocommutative*

$$\hat{A} \hat{\times} \hat{B} \neq \hat{B} \hat{\times} \hat{A}$$

$$A \hat{\times} B \neq B \hat{\times} A$$

but are *isoassociative*, namely:

$$\begin{aligned} \hat{A}\hat{\times}\left(\hat{B}\hat{\times}\hat{C}\right) &= \left(\hat{A}\hat{\times}\hat{B}\right)\hat{\times}\hat{C} \\ A\hat{\times}\left(B\hat{\times}C\right) &= \left(A\hat{\times}B\right)\hat{\times}C \end{aligned} \tag{18}$$

For example, if $\hat{I}^{-1} = \widehat{T} = 4$, $A = 4$ and $B = 5$ then we have $4\hat{\times}5 = 80$ and $\hat{4}\hat{\times}\hat{5} = 5$. This example reveals that the increased secrecy of cryptograms or new cryptograms gets generated using isonumbers [18].

Interestingly, from the above discussion it is easy to arrive at the following *isodefinition* of *isounit*, namely:

$$\widehat{U}\hat{\times}\widehat{U}^{\dagger} = \widehat{U}^{\dagger}\hat{\times}\widehat{U} = \hat{I}$$

under the adoption of $U = \widehat{U} \times \widehat{T}^{1/2}$.

The *isosums* are given as,

$$A\hat{+}B = A + B, \quad \hat{A}\hat{+}\hat{B} = \hat{A} + \hat{B} \quad \text{implying} \quad + = \hat{+} \tag{19}$$

The *isosums* are *isocommutative* and *isoassociative*,

$$\hat{A} + \hat{B} = \hat{B} + \hat{A} \tag{20}$$

$$\left(\hat{A} + \hat{B}\right) + \hat{C} = \hat{A} + \left(\hat{B} + \hat{C}\right) \tag{21}$$

There exists an *additive isounit*, $\hat{0}$, such that,

$$\hat{A} + \hat{0} = \hat{0} + \hat{A} = \hat{A} \tag{22}$$

that is on addition of *isozero* to any *isoquantity* doesn't change its value. Compare this identity with the conventional sums, viz. $A + 0 = 0 + A = A$.

The *isotopiclifting* to *isozero*, $\hat{0}$, is expressed as $0 \longrightarrow U \times 0 \times U^{\dagger} \equiv \hat{0}$.

A *negative isoquantity*, $-\hat{A}$, follows the following relationship with its positive counterpart, namely:

$$\hat{A} + (-\hat{A}) = \hat{0} \tag{23}$$

The isoproducts and isosums follow the right and left *isodistributive law*, namely:

$$\hat{A}\hat{\times}\left(\hat{B} + \hat{C}\right) = \hat{A}\hat{\times}\hat{B} + \hat{A}\hat{\times}\hat{C}, \quad \left(\hat{A} + \hat{B}\right)\hat{\times}\hat{C} = \hat{A}\hat{\times}\hat{C} + \hat{B}\hat{\times}\hat{C} \tag{24}$$

Herein, two possible doubts need to be clarified [1, 9].

We have seen above that the *isoproduct symbol* is given by $\hat{\times} = \times\widehat{T}\times$ hence on the same lines one may expect $\hat{+} = +\widehat{K}+$. If this isotopic lifting is adopted then it would flout the *isodistributive law* of eq.(24). This gets illustrated as follows.

Let us rewrite eq.(24) using $\hat{+}$ that reads as

$$\hat{A}\hat{\times}\left(\hat{B}\hat{+}\hat{C}\right) = \hat{A}\hat{\times}\hat{B}\hat{+}\hat{A}\hat{\times}\hat{C}, \quad \left(\hat{A}\hat{+}\hat{B}\right)\hat{\times}\hat{C} = \hat{A}\hat{\times}\hat{C}\hat{+}\hat{B}\hat{\times}\hat{C} \qquad (25)$$

The left hand side of the first equality, the right isodistributive law, of eq.(25) would transform as,

$$\begin{aligned}\hat{A}\hat{\times}\left(\hat{B}\hat{+}\hat{C}\right) &= A\times\hat{I}\times\widehat{T}\times\left(\hat{B}\hat{+}\hat{C}\right)\\ &= A\times(B+K+C)\times\hat{I} = (A\times B + A\times K + A\times C)\times\hat{I}\end{aligned}$$

whereas the right hand side of same equation would transform as,

$$\begin{aligned}\hat{A}\hat{\times}\hat{B}\hat{+}\hat{A}\hat{\times}\hat{C}, \quad \left(\hat{A}\hat{+}\hat{B}\right)\hat{\times}\hat{C} &= (A\times B)\times\hat{I}\hat{+}(A\times C)\times\hat{I}\\ &= (A\times B)\times\hat{I} + \widehat{K} + (A\times C)\times\hat{I}\\ &= (A\times B + K + A\times C)\times\hat{I}\end{aligned}$$

Thus we see that the two results are not identical, that is it flouts *isodistributive law*. The same is the case with the left isodistributive law. Hence, we have $+ \longrightarrow \hat{+} = +$ in the isoalgebra.

Moreover, one may also consider that $\hat{0} \neq 0$. Let us again consider the isotopic lifting of the identity $A + 0 = A$. This then read as, $\hat{A} + \hat{0} = \hat{A}$. As above if we assume $\hat{+} = +\widehat{K}+$ then we would have,

$$\hat{A}\hat{+}\hat{0} = \hat{A} + \hat{K} + \hat{0} \neq \hat{A}$$

But the identity of eq.(22) would demand that $\hat{0} = -\hat{K}$ that would provide us,

$$\hat{A}\hat{+}\hat{0} = \hat{A} + \hat{K} + \hat{0} = \hat{A} + \hat{K} - \hat{K} = \hat{A}$$

It means that the non-zero value of $\hat{0}$ can be sustained only if $\hat{+} = +\hat{K}+$. But we have seen that the latter identity flouts the *isodistributive law*. Moreover, in this event we will be led to $\hat{0}\hat{\times}\hat{A} \neq \hat{0}$ and $\hat{0}\hat{\times}A \neq \hat{0}$ and hence we

must have $\hat{0} = 0$. Alternatively also the same conclusion follows, namely $\hat{0} = (0 \times \hat{I}) = 0$.

Now it is easy to see that from the above *isotopic liftings* we have,

$$\begin{aligned} \hat{A}^{\hat{n}} &= \hat{A}\hat{\times}\hat{A}\hat{\times}\hat{A}\hat{\times}\cdots\cdots\hat{\times}\hat{A}\hat{\times}\hat{A} \\ &= A \times \hat{I} \times \hat{T} \times A \times \hat{I} \times \hat{T} \times A \times \hat{I} \times \hat{T} \times \cdots\cdots \times \hat{T} \times A \times \hat{I} \\ &= A^n \hat{T}^{n-1} \hat{I}^n = A^n \hat{I} \qquad (26) \\ A^{\hat{n}} &= A\hat{\times}A\hat{\times}A\hat{\times}\cdots\cdots\hat{\times}A \\ &= A \times \hat{T} \times A \times \hat{T} \times A \times \hat{T} \times \cdots\cdots \times \hat{T} \times A \\ &= A^n \left(\hat{T}\right)^{n-1} = A^n \hat{T} \qquad (27) \end{aligned}$$

Notice the difference between the result of multiplication to A^n by $\hat{I}$ and $\hat{T}$.

The *isosquareroot* gets depicted as,

$$\widehat{\hat{A}^{1/2}} = A^{\frac{1}{2}} \times \hat{I}^{1/2} = A^{\frac{1}{2}} \times \hat{I} \qquad (28)$$

But as $\hat{I}$ is the *isounit* we have,

$$\widehat{\hat{I}^{1/2}} = \hat{I} = \hat{I}^{1/2} \qquad (29)$$

that gets illustrated as,

$$\begin{aligned} \widehat{\hat{I}^{1/2}} &= \widehat{\left(a^{\hat{0}}\right)^{1/2}} = \widehat{a^{1/2\hat{\times}\hat{0}}} = a^{\hat{0}} = \hat{I} \qquad (30) \\ \hat{I}^{1/2} &= \left(a^{\hat{0}}\right)^{1/2} = a^{1/2\times\hat{0}} = a^{\hat{0}} = \hat{I} \qquad (31) \end{aligned}$$

Thus the iso- and conventional products of isounit, $\hat{I}$, gets illustrated in general as,

$$\begin{aligned} \hat{I}^{\hat{n}} &= \left(a^{\hat{0}}\right)^{\hat{n}} = a^{\hat{n}\hat{\times}\hat{0}} = a^{\hat{0}} = \hat{I} \qquad (32) \\ \hat{I}^{n} &= \left(a^{\hat{0}}\right)^{n} = a^{n\times\hat{0}} = a^{\hat{0}} = \hat{I} \qquad (33) \end{aligned}$$

where we have used $\hat{0}\hat{\times}\hat{n} = \hat{0} = \hat{0}\hat{\times}n$. Hence we have,

$$\begin{aligned} \left(\hat{I}\right)^n = \left(\hat{I}\right)^{\hat{n}} &= \hat{I}, \quad n = 1, 2, 3, \cdots\cdots \qquad (34) \\ \therefore \left(\hat{T}\right)^n = \left(\hat{T}\right)^{\hat{n}} &= \hat{T}, \quad n = 1, 2, 3, \cdots\cdots \qquad (35) \end{aligned}$$

but we have,

$$\hat{I}^{\hat{0}} = \hat{I} \quad \text{and} \quad \hat{I}^0 = I = 1, \quad \hat{T}^{\hat{0}} = \hat{I} \quad \text{and} \quad \hat{T}^0 = I = 1 \tag{36}$$

Moreover, $\hat{I}$ being an *isounit* there we have,

$$\hat{A}^{\hat{I}} = \hat{A} = A \times \hat{I} \tag{37}$$

(Compare: $A^1 = A$ or $A^I = A$ as $I = +1$).

In view of the above description of *isoalgebra* the reader should try to verify the following.

$$A^{\hat{2}} = A^2\hat{T}, \; A^{-\hat{2}} = A^{-2}\hat{I}^3, \text{ and hence } A^{-\hat{n}} = A^{-n}\hat{T}^{-n-1} = A^{-n}\hat{I}^{n+1}$$

and

$$\hat{A}^{\widehat{2}} = A^2\hat{I}, \; A^{-\widehat{2}} = A^{-2}\hat{I}, \quad \text{and hence} \quad A^{-\hat{n}} = A^{-n}\hat{I}$$

Similarly, verify that,

$$\hat{A}\hat{\times}\hat{B}^{-2} = \left(\frac{A}{B^2}\right) \times \hat{I} = A\hat{\times}B^{-\hat{2}} \quad \text{and} \quad A\hat{\times}B^{-2} = \left(\frac{A}{B^2}\right) \times \widehat{T}$$

The lifting of the conventional division (quotient) to the *isodivision* (*isoquotient*) is shown as,

$$\begin{aligned} A \div B \quad \longrightarrow \quad & U \times (A \div B) \times U^\dagger \\ & = U \times \frac{A}{B} \times U^\dagger \\ & = \frac{U \times A \times U^\dagger}{U \times B \times U^\dagger} \times \left(U \times U^\dagger\right) \\ & = \frac{\hat{A}}{\hat{B}} \times \hat{I} = \frac{A}{B} \times \hat{I} \\ & = \hat{A} \,\hat{\div}\, \hat{B} = \hat{A} \,\hat{/}\, \hat{B} \end{aligned} \tag{38}$$

Thus we have obtained the following nonunitary transform:

$$\div \longrightarrow \hat{\div} = \div \times \hat{I} \quad \text{or} \quad / \longrightarrow \hat{/} = (/) \times \hat{I} \tag{39}$$

The *isoinverse* is represented as,

$$\hat{A}^{-\hat{I}} = \hat{I} \hat{\div} \hat{A}^{\hat{I}} = \frac{\hat{I}}{\hat{A}^{\hat{I}}} \times \hat{I} = \frac{\hat{I}^2}{A \times \hat{I}} = A^{-1} \times \hat{I} \tag{40}$$

Thus we see that $\hat{I}$ being an *isounit* we correctly have $\hat{I} \times \hat{I} = \hat{I}$ and $\hat{I} \div \hat{I} = \hat{I}$.

The isoproduct of an isoquantity with its isoinverse obviously produces isounit, $\hat{I}$, namely:

$$\hat{A}^{\hat{I}} \hat{\times} \hat{A}^{-\hat{I}} = A \times \hat{I} \times \hat{T} \times A^{-1} \times \hat{I} = 1 \times \hat{I} = \hat{I} = \hat{A}^{\hat{0}} \neq I \tag{41}$$

Thus it represents the isorule: anything raised to power $\hat{0}$ is equal to $\hat{I}$. The same remains true for the conventional quantities too, namely:

$$A^{\hat{0}} = \hat{I} = \hat{A}^{\hat{0}} \tag{42}$$

Similarly,

$$\hat{a}^0 = (a \times \hat{I})^0 = a^0 \hat{I}^0 = 1 \times 1 = 1 \tag{43}$$

which is the mathematical expression of the rule: anything raised to power 0 is equal to 1.

The *isotopic lifting* of a *complex number*, say c, is similar to a non-complex number. For example,

$$c = (a + i \times b) \longrightarrow \hat{c} = c \times \hat{I} = (a + i \times b) \times \hat{I} = (\hat{a} + \hat{i} \hat{\times} \hat{b}) \tag{44}$$

where i is the imaginary unit $i = \sqrt{-1}$. Notice that the purely imaginary numbers do not preserve isotopy. For example, $(a + i \times b)$ and $(a - i \times b)$ belong to a set of imaginary numbers but their product is a non-imaginary number, $(a^2 + b^2)$.

Now if $\hat{I} = i$ the isotopy of real numbers is preserved [1]. In this case the isonumber say $\hat{a}$ transforms as,

$$\hat{a} = a \times \hat{I} = ai, \quad \hat{T} = \hat{I}^{-1} = \frac{1}{i} = -i \tag{45}$$

that transforms the isoproduct of isoquantities as,

$$\hat{a} \hat{\times} \hat{b} = (a \times \hat{I}) \times \hat{T} \times (b \times \hat{I}) = (ai)(-i)(bi) = abi = ab\hat{I} = \widehat{ab} \tag{46}$$

and the isodivision of isoquantities transform as,

$$\hat{a}\hat{\div}\hat{b} = a \times \hat{I} \times (\div) \times \hat{I} \times b \times \hat{I} = a \times i \times (\div) \times i \times b \times i = -\frac{a}{b} \times i \quad (47)$$

One can easily verify the validity of all axioms of isotopic algebra in this case.

The *isonorm* is represented as,

$$\hat{|}\hat{A}\hat{|} = | A | \times \hat{I} \quad (48)$$

The *isonorm* of *isoproducts* transform as,

$$\hat{|}\hat{A}\hat{\times}\hat{B}\hat{|} = \hat{|}\hat{A}\hat{|}\hat{\times}\hat{|}\hat{B}\hat{|}$$

2.2 Some of the Isofunctions

The *isotopic lifting* of algebraic functions to an *isofunctions* in the corresponding *isospace* reads as,

$$f(r) \longrightarrow \hat{f}(\hat{r}) = \hat{I} \times f(r) = \hat{I} \times f(\hat{r} \times \hat{T}) \quad (49)$$

and its projection in the original space reads as,

$$f(r \times \hat{T}) \quad (50)$$

Its interpretation is [9] that while isotopically lifting $f(r)$ it preserves its numerical value over the isofield $\hat{F}$ as $\hat{f}(\hat{r})$ on the isospace, $\hat{S}$, because the variable $\hat{r}$ is multiplied by $\hat{T}$ while the unit to which such a variable is referred to is multiplied by the inverse amount $\hat{I} = 1/\hat{T}$. All the numerical differences emerge in the *projection* of $\hat{f}(\hat{r})$ in the original space.

Let us describe the *isofunctions*, logarithmic and exponential ones.

- The *isologarithm* is obtained as,

$$\hat{\log}_{\hat{e}}\hat{a} = \hat{I} \times \log_e a, \quad \hat{\log}_{\hat{e}}\hat{e} = \hat{I}, \quad \hat{\log}_{\hat{e}}\hat{I} = 0 \quad (51)$$

Other *isooperations* on *isologarithms* are,

$$\begin{aligned}
\hat{e}^{\hat{\log}_{\hat{e}}\hat{a}} &= \hat{a} \\
\hat{\log}_{\hat{e}}(\hat{a}\hat{\times}\hat{b}) &= \hat{\log}_{\hat{e}}\hat{a} + \hat{\log}_{\hat{e}}\hat{b} \\
\hat{\log}_{\hat{e}}(\hat{a}\hat{\div}\hat{b}) &= \hat{\log}_{\hat{e}}\hat{a} - \hat{\log}_{\hat{e}}\hat{b} \\
\hat{\log}_{\hat{e}}\left(\hat{a}^{-\hat{I}}\right) &= -\hat{\log}_{\hat{e}}\hat{a} \\
\hat{b}\hat{\times}\hat{\log}_{\hat{e}}\hat{a} &= \hat{\log}_{\hat{e}}\left(\hat{a}^{\hat{b}}\right)
\end{aligned}$$

Now if a is a field quantity then the projection of $\hat{\log}_{\hat{e}}\hat{a}$ on the original space reads as $\log_{\hat{e}} a = \log_e(a \times \hat{T})$.

- The *isoexponentiation* is manipulated as,

$$\begin{aligned}
\hat{e}^{aY} &= \hat{I} + aY/1! + (aY)^{\hat{2}}/2! + \cdots\cdots \\
&= \left[1 + \frac{aY\hat{T}}{1!} + \frac{(aY)\hat{\times}(aY)\times\hat{T}}{2!} + \cdots\cdots\right] \times \hat{I} \\
&= \left[1 + \frac{a\hat{T}Y}{1!} + \frac{(a\hat{T}Y)(a\hat{T}Y)}{2!} + \cdots\cdots\right] \times \hat{I} \\
&= \left[1 + \frac{a\hat{\times}Y}{1!} + \frac{(a\hat{\times}Y)(a\hat{\times}Y)}{2!} + \cdots\cdots\right] \times \hat{I} \\
&= \left(e^{a\hat{\times}Y}\right) \times \hat{I} = \hat{I} \times \left(e^{a\hat{\times}Y}\right) \\
&= \left(e^{a\times\hat{T}\times Y}\right) \times \hat{I} = \hat{I} \times \left(e^{a\times\hat{T}\times Y}\right) \qquad (52)
\end{aligned}$$

Alternatively, one can show the *isoexponentiation* through the isotopic lifting, namely:

$$\begin{aligned}
e^{aY} \rightarrow\ & U \times e^{aY} \times U^{\dagger} = U \times \left(1 + \frac{aY}{1!} + \frac{(aY)^2}{2!} + \cdots\right) \times U^{\dagger} \\
=\ & \hat{I} + \frac{\hat{a}\hat{\times}\hat{Y}}{\hat{1}!} \times \hat{I} + \frac{(\hat{a}\hat{\times}\hat{Y})^{\hat{2}}}{\hat{2}!} \times \hat{I} + \frac{(\hat{a}\hat{\times}\hat{Y})^{\hat{3}}}{\hat{3}!} \times \hat{I} + \cdots\cdots \\
=\ & \hat{e}^{\hat{a}\hat{\times}\hat{Y}} \\
=\ & \hat{I} \times \left(1 + \frac{\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T}}{1!} + \frac{(\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T})^2}{2!} + \cdots\cdots\right) \\
=\ & \hat{I} \times \left(e^{\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T}}\right) = \left(e^{\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T}}\right) \times \hat{I} \\
=\ & \hat{I} \times \left(1 + \frac{aY}{1!} + \frac{(aY)^2}{2!} + \frac{(aY)^3}{3!} + \cdots\right) = \left(e^{aY}\right) \times \hat{I} \qquad (53)
\end{aligned}$$

Thus the projection of $\hat{e}^{\hat{a}\hat{\times}\hat{Y}} = \hat{I} \times \left(e^{\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T}}\right) = \left(e^{\hat{a}\times\hat{T}\times\hat{Y}\times\hat{T}}\right) \times \hat{I}$ on the original space is $\left(e^{a\times\hat{T}\times Y}\right) \times \hat{I} = \hat{I} \times \left(e^{a\times\hat{T}\times Y}\right)$.

2.3 Isotrigonometric and Isohyperbolic Functions

The description of *isotrigonometric* and *isohyperbolic* functions involves the identification of *isospaces* and corresponding *isogeometry* [1,4,9,13,19–21]. Only the relevant aspects are given below keeping the abstract contents to a bare minimum.

2.3.1 Isopythagorean Theorem

The isopythagorian theorem gets illustrated through the computation of the distance between two points in the isospace and the cosine of an isoangle in an isoplane.

Therefore, let us first consider the three dimensional Euclidean space and two points $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$ within it. The distance between these two points, D, is given by:

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{54}$$

which is the Pythagorean theorem. Now the value of D can be reexpressed by using the unit metric, $\delta = \mathrm{diag.}(1, 1, 1) = I$ as,

$$D^2 = \left[(x_i - x_i')\delta_{ij}(x_j - x_j')\right] \quad \text{with } i, j = 1, 2, 3. \tag{55}$$

where now we have represented the points by P and P′.

Next let two vectors originate from the origin, say in the x–y plane, and extend respectively to $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$. Let the angle subtended by two vectors on the x-axis be ϕ and θ respectively and hence the angle between the two vectors is $\alpha(= \phi - \theta)$. Thus using the Pythagoras theorem it is easy to show that the cosine of the angle α depicted in Figure 1 is given by,

$$\cos\alpha = \frac{x_1x_2 + y_1y_2}{(x_1x_1 + y_1y_1)^{1/2}\,(x_2x_2 + y_2y_2)^{1/2}} \tag{56}$$

Now we consider the same two functions D and $\cos\alpha$ in the corresponding *isoeuclidean space*. This isospace is determined by *isoquantities* and hence we have the following *isodefinitions* and *isotopic liftings*,

$$\begin{aligned}
\hat{\delta} = \widehat{T}\,(r, \dot{r}, \ddot{r}, ...)\,\delta = \mathrm{diag.}\left(n_1^{-2},\, n_2^{-2},\, n_3^{-2}\right) \text{ or} \\
\hat{\delta}_{ij} = \widehat{T}_{ik}\delta_{kj} = \widehat{T}_{ii}\delta_{ij} = n_i^{-2}\delta_{ij} \\
\hat{I} = \widehat{T}^{-1} = \mathrm{diag.}\left(n_1^2,\, n_2^2,\, n_3^2\right) \text{ and} \\
n_k = n_k\,(r, \dot{r}, \ddot{r}, ...) > 0 \text{ with } k = 1, 2, 3.
\end{aligned} \tag{57}$$

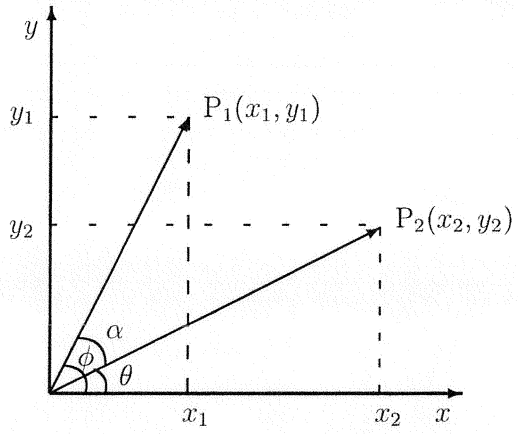


Figure 1: *Computation of* $\sin\alpha$ *and* $\cos\alpha$ *from that of angles* θ *and* ϕ *depicted in this figure*: From the figure we have,

$$\cos\phi = \frac{x_1}{(x_1^2+y_1^2)^{1/2}}, \quad \sin\phi = \frac{y_1}{(x_1^2+y_1^2)^{1/2}},$$
$$\cos\theta = \frac{x_2}{(x_2^2+y_2^2)^{1/2}}, \quad \sin\theta = \frac{y_2}{(x_2^2+y_2^2)^{1/2}}$$

and $\alpha = (\phi - \theta)$. Hence we have $\cos\alpha = \cos(\phi - \theta)$ and $\sin\alpha = \sin(\phi - \theta)$ that enable us to arrive at the expressions of $\cos\alpha$ and $\sin\alpha$ given in the text.

where r is the position vector. Notice that eq.(57) implies that not only the basic conventional multiplying unit, δ, gets isotopically lifted to a nonunitary isounit, $\hat{\delta}$, but also the unit of each Euclidean axis gets isotopically lifted to nonunitary value for each isoaxis and they are not identically same that is we have n_1^2 for $\hat{x}$-isoaxis, n_2^2 for $\hat{y}$-isoaxis and n_3^2 for $\hat{z}$-isoaxis. Of course, we could have adopted identically same values for *isounits* of all *isoaxes* but then we would fall prey to the consideration of only the spherical objects that is we envisage a restricted isospace and corresponding isogeometry.

The geometric implications of the above isotopic liftings of δ and I to the *isometric*, $\hat{\delta}$, and *isounit*, $\hat{I}$, are far reaching. Recall that a given shape in the Euclidean space is determined by the metric δ which on isotopically lifting to $\hat{\delta} = \widehat{T}\delta$ deforms the former because the components of $\widehat{T}$ are not identically same in all directions. But concomitantly the unit of measurement of the amount of a quantity along the isoaxes gets *inversely* lifted to $\hat{I} = \widehat{T}^{-1}$ that perfectly restores back the original geometry. Another way to explain it is — on the isotopic lifting the ordinary metric is arbitrarily de-

formed, $\delta \longrightarrow \hat{\delta} = \widehat{T}\delta$, thus acquiring the most general possible curvature, but at the same time the original unit is deformed by the inverse amount $I \longrightarrow \hat{I} = \widehat{T}^{-1}$, thus eliminating the acquired curvature in the isospace. Hence, we rigorously have the following isotopic liftings,

$$
\begin{aligned}
\textit{straigt lines} &\longrightarrow \textit{isostraight isolines} \\
\textit{circle} &\longrightarrow \textit{isocircle} \\
\textit{triangle} &\longrightarrow \textit{isotriangle (with rectilinear sides)} \\
\textit{right triangle} &\longrightarrow \textit{isoright isotriangle (with rectilinear sides)} \\
\textit{angle} &\longrightarrow \textit{isoangle} \\
\textit{and so on} &\ldots\ldots\ldots
\end{aligned}
$$

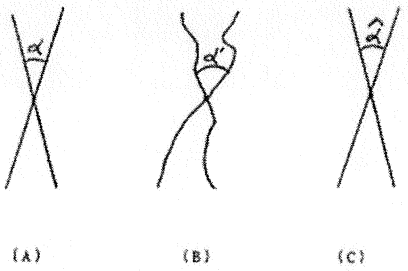


Figure 2: *Isoangle* and *Isostraight isolines*: Diagram (A) depicts the origin of the notion of angle, shown here by α, in the Euclidean plane from two intersecting straight lines. Diagram (B) depicts the effect of the transformation $\delta \longrightarrow \hat{\delta} = \widehat{T}\delta$. Notice that the original straight lines get transformed into curves and hence the traditional meaning of an angle gets lost (Herein the angle formed by the tangent vectors at the point of intersection is taken as the definition of an angle formed by two intersecting curves which obviously will have different value than that in the original Euclidean plane.). Next, diagram (C) shows the further transformation on account of $I \longrightarrow \hat{I} = \widehat{T}^{-1}$. Because I gets deformed to $\hat{I}$ by an inverse amount the curves of diagram (B) get transformed back to straight lines, termed as *isostraight isolines*, and hence again the traditional definition of angle gets restored. As this angle belongs to the *isoeuclidean plane* it is represented by $\hat{\alpha}$ and is termed as *isoangle*.

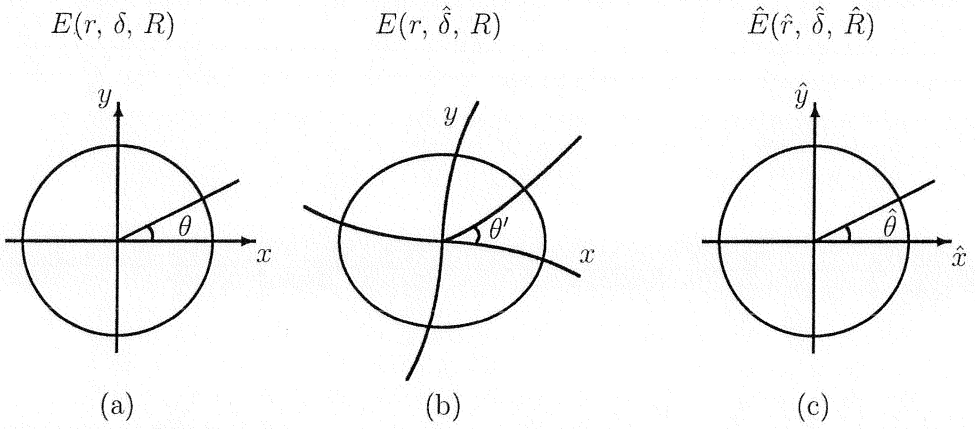


Figure 3: *Isocircle*: Diagram (a) depicts the circle in traditional Euclidean plane and the angle, θ, formed by a vector originating from the origin with the x-axis. Diagram (b) depicts the deformation of the axes and circle to an elliptic shape due to the deformation $\delta \longrightarrow \hat{\delta} = \widehat{T}\delta$ and also the deformed angle θ' now formed by the intersecting tangent vectors originating from the origin. Next diagram (c) depicts the restoration back of the circle, now termed as *isocircle*, effected by the deformation $I \longrightarrow \hat{I} = \widehat{T}^{-1}$ and also the restoration of the traditional definition of angle in *isoplane*, which is now termed as *isoangle*, $\hat{\theta}$.

In other words the adoption of $\delta \longrightarrow \hat{\delta} = \widehat{T}\delta$ and $I \longrightarrow \hat{I} = \widehat{T}^{-1}$ is necessary and sufficient to preserve all the axioms of the Euclidean geometry on the isotopic liftings. This is depicted below in Figure 2 and Figure 3 for straight lines, an angle and a circle.

Notice that the metric $\hat{\delta}$ defined in eq.(57) is not an isonumber. For this purpose the correct definition of the *isometric* is given by [9],

$$\hat{\Delta} = \hat{\delta} \times \hat{I} \tag{58}$$

which would be used hereafter.

2.3.2 Isoeuclidean Distance Between Two Isopoints

Thus in isoeuclidean space the *isoeuclidean distance* between two points $\hat{\mathrm{P}}(\hat{x})$ and $\hat{\mathrm{P}}'(\hat{x}')$ on using the *isometric* of eq.(58) is given by,

$$\begin{aligned}\hat{D}^{\hat{2}} &= \left[(\hat{x}_i - \hat{x}'_i)\hat{\times}\hat{\Delta}_{ij}\hat{\times}(\hat{x}_j - \hat{x}'_j)\right] \\ &= \left[(\hat{x}_i - \hat{x}'_i)\hat{\delta}_{ij}(\hat{x}_j - \hat{x}'_j) \times \hat{T}\right] \qquad \text{with } i,j = 1,2,3. \end{aligned} \tag{59}$$

which is the corresponding *isopythagorean theorem*. Indeed, we can further manipulate eq.(59) by using eq.(57) that reads as,

$$\begin{aligned}\hat{D}^{\hat{2}} &= \left[(\hat{x}_i - \hat{x}'_i)\hat{\delta}_{ij}(\hat{x}_j - \hat{x}'_j) \times \hat{T}\right] \\ &= \left[(\hat{x}_i - \hat{x}'_i)n_i^{-2}\delta_{ij}(\hat{x}_j - \hat{x}'_j) \times \hat{T}\right] \\ &= \left[(\hat{x}_i - \hat{x}'_i)n_i^{-2}(\hat{x}_i - \hat{x}'_i) \times \hat{T}\right] \\ &= \left[(x_i - x'_i)n_i^{-2}(x_i - x'_i)\right] \times \hat{I} \end{aligned} \tag{60}$$

Thus we see that the multiplication by $\hat{I}$ on the right hand side of the last equality of eq.(60) guarantees that $\hat{D}$ belongs to the isoeuclidean space.

In spite of this visible difference between D and $\hat{D}$, all conventional notions in Euclidean space get preserved under isotopies provided that they are computed in the isoeuclidean space. In this way, as stated above we have the notions of *isolines*, *isostraight isolines*, *isotriangle*, *isoangle*, *isoright-isotriangle*, etc.

From eq.(60) the concept of projection on original Euclidean space originates. The said projection is now given by,

$$D = \left|\left[(x_i - x'_i)n_i^{-2}(x_i - x'_i)\right]^{1/2}\right| \tag{61}$$

Notice that it is not the identically same distance that we have directly calculated before isotopic lifting, viz. eqs.(54) or (55).

Thus the *isopythagorean theorem* reads as:

The following property holds in the isoeuclidean plane,

$$\hat{D}^{\hat{2}} = \hat{D}\hat{\times}\hat{\triangle}\hat{\times}\hat{D} = \hat{A}^{\hat{2}} + \hat{B}^{\hat{2}} = \hat{A}\hat{\times}\hat{\triangle}\,\hat{\times}\hat{A} + \hat{B}\hat{\times}\hat{\triangle}\hat{\times}\hat{B} \tag{62}$$

which on using isotopic algebra gives,

$$\hat{D}^{\hat{2}} = D^2 \times \hat{I} = \left[A\, n_1^2\,(r, \dot{r}, \ddot{r}, ...)\; A + B\, n_2^2\,(r, \dot{r}, \ddot{r}, ...)\; B\right] \times \hat{I} \quad (63)$$

that is the isosquare of isohypotenuse of an isoright-isotriangle is the sum of the isosquares of isosides. Also notice that the components of *isounit*, n_i^2, do not get reduced to unity by the above said projection on the original Euclidean space. Hence the symbol D of eqs.(61) and (63) doesn't describe the distance represented by the symbol D of eq.(54) or (55). To remove this ambiguity there is a need to make a distinction between the two descriptions. Fortunately it is possible and the result of this adoption is profitable, that we will see below.

Further, from eq.(63) we get the following expression of the projection on original Euclidean plane, namely:

$$D^2 = \left[A\, n_1^2\,(r, \dot{r}, \ddot{r}, ...)\; A + B\, n_2^2\,(r, \dot{r}, \ddot{r}, ...)\; B\right] \quad (64)$$

Thus we see that the value of the distance obtained on projection in the original Euclidean plane is not the same because before isotopic lifting the distance was given by $D^2 = A^2 + B^2$.

At this stage of the development it is essential to understand that there are three different interpretations of the *isopythagorean theorem* as there are corresponding three Euclidean spaces. The first is the distance calculated in isospace, $\hat{E}(\hat{r},\, \hat{\delta},\, \hat{R})$, where $\hat{r}$ is the *isoradius isovector* and $\hat{R}$ denotes a set of *isonumbers*, its example are eqs.(59) or (60) and (62). The second is via the projection on the *original Euclidean space*, $E(r,\, \delta,\, R)$, its example are the distance computed in eqs.(61) and (64). The third option is when we identify an Euclidean space as, $\bar{E}(\bar{r},\, \delta,\, R)$ where the radius vector, $\bar{r}$, is computed by defining new coordinates as,

$$\begin{aligned} &\bar{x} = x n_1^{-1}(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, ...), \qquad \bar{y} = y n_2^{-1}(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, ...), \\ &\bar{z} = z n_3^{-1}(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, ...) \end{aligned} \quad (65)$$

Before, using these new definitions of eq.(65) let us re-manipulate the steps

of eq.(60), that reads as,

$$\begin{aligned}\hat{D}^{\hat{2}} &= \hat{D}\hat{\times}\hat{\triangle}\hat{\times}\hat{D} = (\hat{D}\times\hat{\delta}\times\hat{D})\times\hat{T} = \hat{D}^2\times\hat{T} \\ &= (\hat{x}_i - \hat{x}'_i)\hat{\delta}_{ij}(\hat{x}_j - \hat{x}'_j)\times\hat{T} \\ &= (\hat{x}_i - \hat{x}'_i)\hat{\delta}_{ii}(\hat{x}_i - \hat{x}'_i)\times\hat{T} \\ &= (\hat{x}_i - \hat{x}'_i)n_i^{-2}(\hat{x}_i - \hat{x}'_i)\times\hat{T} \end{aligned} \tag{66}$$

and hence we have,

$$\hat{D}^2 = (\hat{x}_i - \hat{x}'_i)n_i^{-2}(\hat{x}_i - \hat{x}'_i) = [(x_i - x'_i)n_i^{-2}(x_i - x'_i)]\times\hat{I}^2 \tag{67}$$

Now substitute eq.(65) in the last equality of eq.(67) that yields,

$$\hat{D}^2 = [(\bar{x}_i - \bar{x}'_i)\times(\bar{x}_i - \bar{x}'_i)]\times\hat{I}^2 \tag{68}$$

that in expanded version reads as,

$$\hat{D}^2 = \left[(\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2 + (\bar{z}_2 - \bar{z}_1)^2\right]\times\hat{I}^2 \tag{69}$$

From eq.(67) we obtain the projection on the *conventional Euclidean space*, $E(\bar{r},\ \delta,\ R)$, of the distance between two points of corresponding *isoeuclidean space*, as:

$$\bar{D}^2 = (\bar{x}_i - \bar{x}'_i)\times(\bar{x}_i - \bar{x}'_i) \tag{70}$$

Notice that eq.(70) resembles with eq.(54) just because the elements of $\hat{\delta}$ have been absorbed in the corresponding Euclidean coordinate system but it retains the overall iso-character of the distance computed. In the view of the latter the distance interval computed say from eq.(69) coincides with that given by eq.(59).

Notice that on using the variables $\bar{x}_i$'s we conveniently use the conventional multiplication (or division) but still the result on multiplying the corresponding expression by the isounit is the original isoquantity of isospace (c.f. eqs. (67) and (69)). The advantage of the representation is that when a quantity is the isodivision of two isoquantities we can directly describe it by using $\bar{x}_i$'s using traditional multiplication and division.

Thus all isotopies preserve all original properties of the given geometry that has been isotopically lifted. In this sense, a right triangle in the conventional Euclidean plane remains so in the isoplane as a *isoright isotriangle*, and the conventional Pythagorean Theorem holds also in the isospace

in the form of *isopythagorean theorem*. Thus, when an Euclidean geometry (straight line, triangle, circle, etc.) is isotopically lifted it preserves the geometry in the isospace. But the projection of an isogeometry back on the original Eulidean and the conventional Euclidean spaces may not preserve the geometry. For example,

- an *isostraight-isoline* say,

$$\hat{a}\hat{\times}\hat{x} + \hat{b}\hat{\times}\hat{y} + \hat{c}\hat{\times}\hat{z} + \hat{d} = \hat{0} \tag{71}$$

  when projected on $E(r,\ \delta,\ R)$ does map a straight line, namely:

$$ax + by + cz + d = 0 \tag{72}$$

  but when isostraight-isoline of eq.(71) is projected on the conventional iso-euclidean space, $\bar{E}(\bar{r},\ \delta,\ R)$, it maps a curve, namely:

$$a\times\bar{x}\times n_1(r, \dot{r}, ...) + b\times\bar{y}\times n_2(r, \dot{r}, ...) + c\times\bar{z}\times n_3(r, \dot{r}, ...) + d = 0. \tag{73}$$

  this is so because isounits, which are the functions of local variables, do not get mapped into the conventional unit.

- when *isoright-isotriangle* is projected on $\bar{E}(\bar{r},\ \delta,\ R)$ it maps a geometric figure of which the sides are curved with only one point of intersection per pair as depicted in Figure 4. That is the geometry is not preserved.

- on projecting an *isocircle* on $\bar{E}(\bar{r},\ \delta,\ R)$ preserves the geometry but projecting on $E(r, \delta, R)$ it does not

$$r^2 = \frac{x^2}{n_1^2} + \frac{y^2}{n_2^2}. \tag{74}$$

  This is shown in Figure 5 and described below through mathematical transformation. The following mathematical manipulations proves the above assertions. Let us consider an *isoeuclidean plane* with *isoaxes* $\hat{x}$ and $\hat{y}$. The distance of a point P$(\hat{x},\ \hat{y})$ lying in this *isoplane* from the origin $(\hat{0},\ \hat{0})$ is denoted by the *isoradius isovector* $\hat{r}$. Hence, from eq.(59) or eq.(66) we have,

$$\hat{r}^{\hat{2}} = \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} \tag{75}$$

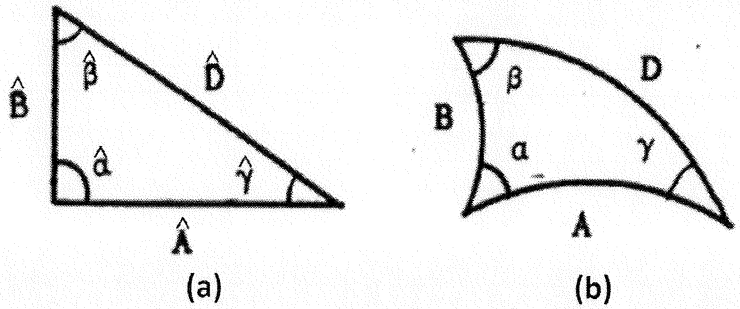


Figure 4: *Isoright-isotriangle*: Diagram (a) depicts the *isoright-isotriangle* with sides $\hat{\mathrm{A}}$, $\hat{\mathrm{B}}$ and $\hat{\mathrm{D}}$, and angles $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ whereas diagram (b) depicts the projection in $E(\bar{r},\, \delta,\, R)$ of the *isoright-isotrangle.* Notice that in diagram (B) not only the sides are curves its angles are different than the corresponding isoangles.

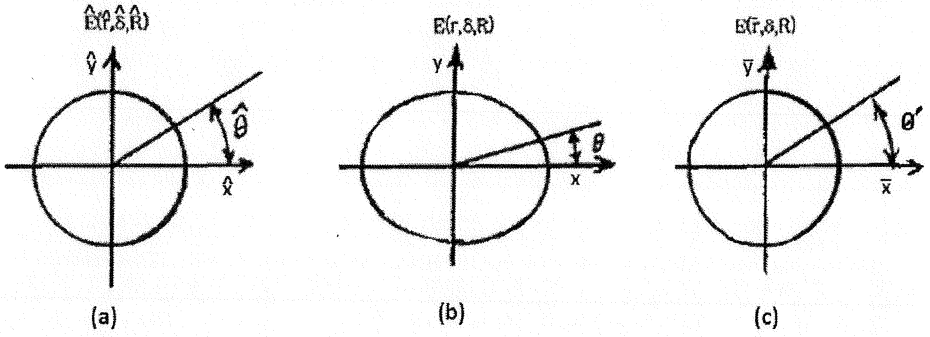


Figure 5: *Projection of an Isocircle*: Diagram (a) depicts an *isocircle* in *isoeuclidean plane.* The projection of the isocircle on the original Euclidean plane shown in diagram (b) is an elliptic shape, that is the geometry is not preserved. Whereas diagram (c) is the projection on the conventional Euclidean plane, $E(\bar{r},\, \delta,\, R)$, wherein the original geometry is preserved.

which is an equation of an *isocircle* and thus we see that on *isotopic* lifting preserves the geometry. However, in this case from eq.(68) we

have,

$$\hat{r}^2 = (\bar{x}^2 + \bar{y}^2) \times \hat{I} \tag{76}$$

and hence we obtain,

$$\bar{r}^2 = (\bar{x}^2 + \bar{y}^2) \tag{77}$$

which is the projection of an *isocircle* on $\bar{E}(\bar{r},\, \delta,\, R)$ and we see that it is an equation of a circle. Thus in *conventional Euclidean plane* the geometry is preserved. On the other hand, eq.(77) on using the definitions of eq.(65) reads as,

$$\bar{r}^2 = r^2 = \frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} \tag{78}$$

which is the equation of an ellipse but not that of a circle.

2.3.3 Isocosine and Isosine of an Isoangle

The preservation of the geometry of *right triangle* on isotopically lifting allows us to isotopically lift the expression of the cosine of an angle, eq.(56).

Let us use the above described convention according to which the symbols $\hat{\alpha}$, $\hat{x}$, $\hat{y}$, etc., denote the quantities computed in isospace, $\hat{E}(\hat{r},\, \hat{\delta},\, \hat{R})$, the symbols $\bar{\alpha}$, $\bar{x}$, $\bar{y}$, etc., denote the corresponding quantities when computed in the conventional Euclidean space $E(\bar{r},\, \delta,\, R)$, and the symbols α, x, y, etc., denotes the projection in the conventional space $E(r,\, \delta,\, R)$.

Suppose that the two points $\hat{P}(\hat{x}_1,\, \hat{y}_1)$ and $\hat{P}(\hat{x}_2,\, \hat{y}_2)$ represent the tips of the respective *isostraight isovectors* initiating from the origin $\hat{0}\, \epsilon\, \hat{E}(\hat{r},\, \hat{\delta},\, \hat{R})$. Let us denote by $\hat{\alpha}$ the *isoangle* between these two isovectors. Since the cosine of an angle is given by eq.(56) and because on isotopic lifting the right triangle geometry is preserved as isoright isotriangle the corresponding expression of isocosine of and isoangle would be given similarly by using the corresponding isoquantities. But as stated above this can also be conveniently expressed using $\bar{x}_i$'s and $\bar{y}_i$'s. Keeping this in view the expression of *isocosine* is given by,

$$\widehat{\cos}\, \hat{\alpha} = \frac{\bar{x}_1 \times \bar{x}_2 + \bar{y}_1 \times \bar{y}_2}{(\bar{x}_1 \times \bar{x}_1 + \bar{y}_1 \times \bar{y}_1)^{1/2} \times (\bar{x}_2 \times \bar{x}_2 + \bar{y}_2 \times \bar{y}_2)^{1/2}} \times \hat{I} \tag{79}$$

and on using the definitions of eq.(65) in eq.(79) we obtain,

$$\cos\, \hat{\alpha} = \frac{x_1 n_1^{-2} x_2 + y_1 n_2^{-2} y_2}{\left(x_1 n_1^{-2} x_1 + y_1 n_2^{-2} y_1\right)^{1/2} \times \left(x_2 n_1^{-2} x_2 + y_2 n_2^{-2} y_2\right)^{1/2}} \tag{80}$$

which, one can see, is an ordinary scalar (rather than the isoscalar).

Now it is possible to generate the expression of *isocosine* in terms of isoquantities from eq.(79). First it is easy to show that we have,

$$\begin{aligned}
(\bar{x}_1 \times \bar{x}_1 + \bar{y}_1 \times \bar{y}_1)^{1/2} \times \hat{I}^{\widehat{1/2}} &= \left(\hat{x}_1 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_1 + \hat{y}_1 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \bar{y}_1\right)^{\widehat{1/2}} \\
(\bar{x}_2 \times \bar{x}_2 + \bar{y}_2 \times \bar{y}_2)^{1/2} \times \hat{I}^{\widehat{1/2}} &= \left(\hat{x}_2 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_2 + \hat{y}_2 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \bar{y}_2\right)^{\widehat{1/2}} \\
(\bar{x}_1 \times \bar{x}_2 + \bar{y}_1 \times \bar{y}_2) \times \hat{I} &= \left(\hat{x}_1 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_2 + \hat{y}_1 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \bar{y}_2\right) \qquad (81)
\end{aligned}$$

where $\hat{\triangle}_{xx} = \hat{I} \times \hat{\delta}_{xx}$ and $\hat{\triangle}_{yy} = \hat{I} \times \hat{\delta}_{yy}$. Thus on using above expressions eq.(79) gets expressed as,

$$\cos\hat{\alpha} = \frac{\left(\hat{x}_1 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_2 + \hat{y}_1 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \hat{y}_2\right)}{\left(\hat{x}_1 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_1 + \hat{y}_1 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \hat{y}_1\right)^{\widehat{1/2}} \times \left(\hat{x}_2 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_2 + \hat{y}_2 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \hat{y}_2\right)^{\widehat{1/2}}} \qquad (82)$$

This expression we can write directly too. Further, from Figure 1 we have the following expression of $\sin\alpha$, namely:

$$\sin\alpha = \frac{y_1 x_2 - x_1 y_2}{(x_1 x_1 + y_1 y_1)^{1/2} (x_2 x_2 + y_2 y_2)^{1/2}}. \qquad (83)$$

from which we directly write the following expression of *isosine* of *isoangle alpha*,

$$\widehat{\sin}\hat{\alpha} = \frac{\bar{y}_1 \times \bar{x}_2 - \bar{x}_1 \times \bar{y}_2}{(\bar{x}_1 \times \bar{x}_1 + \bar{y}_1 \times \bar{y}_1)^{1/2} \times (\bar{x}_2 \times \bar{x}_2 + \bar{y}_2 \times \bar{y}_2)^{1/2}} \times \hat{I} \qquad (84)$$

which produces the following expression of $\sin\hat{\alpha}$,

$$\sin\hat{\alpha} = \frac{y_1 (n_2^{-1} n_1^{-1}) x_2 - x_1 (n_1^{-1} n_2^{-1}) y_2}{\left(x_1 n_1^{-2} x_1 + y_1 n_2^{-2} y_1\right)^{1/2} \times \left(x_2 n_1^{-2} x_2 + y_2 n_2^{-2} y_2\right)^{1/2}} \qquad (85)$$

which too is a normal scalar. Now to generate the expression of $\sin\hat{\alpha}$ in terms of corresponding isoquantities the manipulation of denominator is the same as that we have adopted in the case of $\cos\hat{\alpha}$ but for the numerator

we have to adopt a different values of isomultiplication for $\hat{x}_i$'s and $\hat{y}_i$'s. Thus we adopt the following,

$$\hat{\times}_x = n_1^{-2},\ \hat{\times}_y = n_2^{-2} \tag{86}$$

that yields,

$$\hat{\triangle}_{xy} = \hat{\triangle}_{yx} = n_2^{-1}\, n_1^{-1} = n_1^{-1}\, n_2^{-1} \tag{87}$$

With these adoptions eq.(85) gets expressed as,

$$\sin\hat{\alpha} = \frac{\hat{y}_1 \hat{\times}_y \hat{\triangle}_{yx} \hat{\times}_x \hat{x}_2 - \hat{x}_1 \hat{\times}_x \hat{\triangle}_{xy} \hat{\times}_y \hat{y}_2}{\left(\hat{x}_1 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_1 + \hat{y}_1 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \hat{y}_1\right)^{\widehat{1/2}} \times \left(\hat{x}_2 \hat{\times} \hat{\triangle}_{xx} \hat{\times} \hat{x}_2 + \hat{y}_2 \hat{\times} \hat{\triangle}_{yy} \hat{\times} \hat{y}_2\right)^{\widehat{1/2}}} \tag{88}$$

Thus we see that the metric for multiplications of quantities belonging to the same axis is not the same for the multiplication of quantities belonging to different axes. That is $\hat{\triangle}_{xy} \neq \hat{\triangle}_{xx} \neq \hat{\triangle}_{yy}$. This reveals that in isospaces we have different *isounits* and *isotopicmetrics*.

2.3.4 Isotopy of Isoangles

Now from eq.(85) we attempt to derive the multiplicative *isounit of an angle*. Consider an *unit isocircle* and two isovectors of unit isolength and subtending isoangles $\hat{\phi}$ and $\hat{\theta}$ with the $\hat{x}$ axis. In view of this assumption the terms in the denominator of eq.(85) become unity individually, namely:

$$\begin{aligned} x_1 n_1^{-2} x_1 + y_1 n_2^{-2} y_1 &= 1 \\ x_2 n_1^{-2} x_2 + y_2 n_2^{-2} y_2 &= 1 \end{aligned} \tag{89}$$

that reduces eq.(85) to,

$$\begin{aligned} \sin\hat{\alpha} &= y_1 (n_2^{-1} n_1^{-1}) x_2 - x_1 (n_1^{-1} n_2^{-1}) y_2 \\ &= (y_1 \times x_2 - x_1 \times y_2)\,(n_1^{-1} n_2^{-1}) \end{aligned} \tag{90}$$

However, from eq.(83) we see that for a unit circle with two vectors of unit length we have $\sin\alpha = y_1 \times x_2 - x_1 \times y_2$, hence, eq.(90) becomes,

$$\sin\hat{\alpha} = (n_1^{-1} n_2^{-1}) \sin\alpha \tag{91}$$

In the next step we impose the limit of angle tends to zero on eq.(91) that gives,

$$\hat{\alpha} = (n_1^{-1} n_2^{-1})\, \alpha \tag{92}$$

Though this derivation is not the most general one but the result obtained is perfectly general. Thus we see that the *angular isotopic element*, $\hat{T}_\alpha = (n_1^{-1} n_2^{-1})$, and hence eq.(92) get expressed as $\hat{\alpha} = \hat{T}_\alpha \times \alpha$. And the correspondingly the *angular isounit* is given by,

$$\hat{I}_{\hat{\alpha}} = n_1 n_2 \tag{93}$$

Notice that in terms of spherical coordinate systems the angle, α, in x-y plane considered above is the azimuthal angle. Thus above derived expressions of $\hat{T}_{\hat{\alpha}}$ and $\hat{I}_{\hat{\alpha}}$ are for the *isoazimuthal angle*. Now consider the polar angle subtended by the position vector r with the z-axis which by definition is measured clockwise. Let this angle be denoted by θ. The corresponding three dimensional Euclidean space is $E(r,\, \delta,\, R)$. The cosine of this angle is given by,

$$\cos\theta = \frac{z}{r} = \sin\left(\frac{\pi}{2} - \theta\right) \tag{94}$$

where $r(x,\, y,\, z)$ expresses r in terms of corresponding cartesian coordinates.

Now on isotopic lifting of the polar angle θ lead us to the *isopolar angle*, $\hat{\theta}$ and r gets lifted to the *isospherical coordinates* $\hat{r}(\hat{x},\, \hat{y},\, \hat{z})$ of *isoeuclidean* three dimensional space, $\hat{E}(\hat{r},\, \hat{\delta},\, \hat{R})$. The correspondingly *the isocosine in terms of isopolar angle* is now given by,

$$\widehat{\cos}\,\hat{\theta} = \frac{\hat{z}}{\hat{r}} \times \hat{I} = \widehat{\sin}\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\theta}\right) \tag{95}$$

or else we have,

$$\cos\hat{\theta} = \frac{\hat{z}}{\hat{r}} = \sin\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\theta}\right) \tag{96}$$

Now as $\hat{I} = Diag.(n_1^2,\, n_1^2,\, n_1^2)$ we have $\hat{z} = z n_3^2$ and the *isounit length* on z-axis measures n_3^3. Therefore, we have from eqs.(94) and (97),

$$\cos\hat{\theta} = \frac{z}{r} \times n_3^{-1} = \sin\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\theta}\right) = n_3^{-1} \times \sin\left(\frac{\pi}{2} - \theta\right) \tag{97}$$

Next impose the limit $\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\theta}\right)$ and $\left(\frac{\pi}{2} - \theta\right)$ tend to zero on eq.(97) and we obtain,

$$\widehat{\left(\frac{\pi}{2}\right)} - \hat{\theta} = n_3^{-1} \times \left(\frac{\pi}{2} - \theta\right) \tag{98}$$

or else in general the *isopolar* and polar angles are related as,

$$\hat{\theta} = n_3^{-1} \times \theta \tag{99}$$

This then means that *angular isotopic element* of the polar angle is given by $\hat{T}_{\hat{\theta}} = n_3^{-1}$ and the *angular isounit* of the polar angle is given by $\hat{I}_{\hat{\theta}} = n_3$. Thus it gets once again established that in *isospace* we have different matric for computing different *isoquantities*.

However, for the rigorous derivation of above angular isotopic elements and isounits the reader is advised to refer [1, 11, 22, 23].

The implications of eq.(92) are to be understood with reference to the Figure (4). The figure (a) in Figure (4) is an *isoright-isotriangle*. Hence we have $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = \hat{\pi}$. And on lifting $\alpha = 90^0 \rightarrow \hat{\alpha} = \widehat{\left(\frac{\pi}{2}\right)}$ the value of the angle is preserved. But the projection back in the original plane of $\hat{\alpha} = \widehat{\left(\frac{\pi}{2}\right)}$ does not yield the angle $\alpha = 90^0$, instead the obtained angle α follows the relation $\hat{\alpha} = (n_1^{-1} n_2^{-1})\, \alpha$. Hence, the angles of figure (b) of Figure (4) do not sum to π that is $\alpha + \beta + \gamma \neq 180^0$. However, we do have $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = (n_1^{-1} n_2^{-1})(\alpha + \beta + \gamma) = \hat{\pi}$.

2.3.5 Isotrigonometric Identities

It is then easy to see that the isotrigonometric functions are periodic as in the conventional case, namely:

$$\begin{aligned} \widehat{\cos}(\hat{\alpha} + 2k\pi) &= \widehat{\cos}\,\hat{\alpha} \\ \widehat{\sin}(\hat{\alpha} + 2k\pi) &= \widehat{\sin}\,\hat{\alpha}, \qquad k = 1,\, 2,\, 3, ... \end{aligned} \tag{100}$$

and also,

$$\widehat{\cos}\,(-\hat{\alpha}) = \widehat{\cos}\,\hat{\alpha}, \quad \widehat{\sin}\,(-\hat{\alpha}) = -\widehat{\sin}\,\hat{\alpha} \tag{101}$$

Next the *theorems of Isoadditions of Isotrigonemetric functions* gets derived from eqs.(82) and (88).

From above relations we obtain the following *isotrigonometric* identities,

$$\widehat{\cos}(\hat{\phi} \pm \hat{\theta}) = n_1 \left(n_1^{-2} \widehat{\cos}\,\hat{\phi}\ \widehat{\cos}\,\hat{\theta} \mp n_2^{-2} \widehat{\sin}\,\hat{\phi}\ \widehat{\sin}\,\hat{\theta} \right) \tag{102}$$

$$\widehat{\sin}(\hat{\phi} \pm \hat{\theta}) = n_1^{-1} \left(\widehat{\sin}\,\hat{\phi}\ \widehat{\cos}\,\hat{\theta} \pm \widehat{\cos}\,\hat{\phi}\ \widehat{\sin}\,\hat{\theta} \right) \tag{103}$$

and

$$\widehat{\cos}(2\,\hat{\alpha}) = n_1 \left(n_1^{-2} \widehat{\cos}^{\,2}\hat{\alpha} - n_2^{-2} \widehat{\sin}^{\,2}\hat{\alpha} \right), \quad \widehat{\sin}(2\,\hat{\alpha}) = 2 n_1^{-1} \widehat{\cos}\,\hat{\alpha}\ \widehat{\sin}\,\hat{\alpha} \tag{104}$$

From eqs.(102) - (104) we obtain,

$$\begin{aligned} \widehat{\sin}\hat{\alpha} + \widehat{\sin}\,\hat{\beta} &= 2 n_1^{-1} \widehat{\sin} \left(\frac{\hat{\alpha} + \hat{\beta}}{2} \right) \widehat{\cos} \left(\frac{\hat{\alpha} - \hat{\beta}}{2} \right) \\ \widehat{\sin}\hat{\alpha} - \widehat{\sin}\,\hat{\beta} &= 2 n_1^{-1} \widehat{\cos} \left(\frac{\hat{\alpha} + \hat{\beta}}{2} \right) \widehat{\sin} \left(\frac{\hat{\alpha} - \hat{\beta}}{2} \right) \end{aligned} \tag{105}$$

Other *isotrigonometric* identities can be worked out.

2.3.6 Isoparallelity and Isoperpendicularity of Isostraight Isolines

From eqs.(80) and (85) we have following conditions. The *isoperpendicular isovectors* subtend an *isoangle* between them of $\widehat{\left(\frac{\pi}{2}\right)}$, that from eq.(80) gives the following condition of *isoperpendicularity of isovectors*, namely:

$$x_1 n_1^{-2} x_2 + y_1 n_2^{-2} y_2 = 0 \tag{106}$$

or equivalently,

$$\bar{x}_1 \bar{x}_2 + \bar{y}_1 \bar{y}_2 = 0 \tag{107}$$

and if the *isoangle* between two *isovectors* is $\hat{0}^0$ they are the *isoparallel isovectors*. Thus from eq.(85) we obtain the following condition of *isoparallel isovectors*, namely:

$$y_1 (n_2^{-1} n_1^{-1}) x_2 - x_1 (n_1^{-1} n_2^{-1}) y_2 = 0 \tag{108}$$

and equivalently,

$$\bar{y}_1 \bar{x}_2 - \bar{x}_1 \bar{y}_2 = 0 \tag{109}$$

2.3.7 Isotrigonometry via Isoexponentiation

Isotrigonometric functions also follows via *isoexponentiation*. The *isoexponentiation* gives [22,23],

$$\begin{aligned}
\hat{e}^{i\hat{\phi}} &= \hat{I}_{\hat{\phi}} \times e^{i\widehat{T}\phi} \\
&= \hat{I}_{\hat{\phi}} \times \left(1 + \frac{i\widehat{T}\phi}{1!} + \frac{(i\widehat{T}\phi)^2}{2!} + \frac{(i\widehat{T}\phi)^3}{3!} + \cdots\cdots\right) \\
&= \hat{I}_{\hat{\phi}} \times \left[1 + i\frac{1}{1!}(\hat{T}_{\hat{\phi}}\phi) - \frac{1}{2!}(\hat{T}_{\hat{\phi}}\phi)^2 - i\frac{1}{3!}(\hat{T}_{\hat{\phi}}\phi)^3 \right. \\
&\qquad \left. + \frac{1}{4!}(\hat{T}_{\hat{\phi}}\phi)^4 + i\frac{1}{5!}(\hat{T}_{\hat{\phi}}\phi)^5 - \frac{1}{6!}(\hat{T}_{\hat{\phi}}\phi)^6 - i\frac{1}{7!}(\hat{T}_{\hat{\phi}}\phi)^7 + \cdots\cdots\right] \\
&= \hat{I}_{\hat{\phi}} \times \left(\cos\hat{\phi} + i\sin\hat{\phi}\right) = n_1 n_2 \times \left(n_1^{-1}\widehat{\cos}\hat{\phi} + i n_2^{-1}\widehat{\sin}\hat{\phi}\right) \\
&= n_2\,\widehat{\cos}\,\hat{\phi} + i\,n_1\,\widehat{\sin}\,\hat{\phi} \qquad (110)
\end{aligned}$$

where in the last but one step we have used the definition $n_1 \cos\hat{\alpha} \stackrel{\text{def}}{\equiv} \widehat{\cos}\,\hat{\alpha}$ and $n_2 \sin\hat{\alpha} \stackrel{\text{def}}{\equiv} \widehat{\sin}\,\hat{\alpha}$. Hence similarly we have,

$$\begin{aligned}
\hat{e}^{i\hat{\theta}} &= n_2\,\widehat{\cos}\,\hat{\theta} + i\,n_1\,\widehat{\sin}\,\hat{\theta}, \quad \hat{e}^{-i\hat{\theta}} = n_2\,\widehat{\cos}\,\hat{\theta} - i\,n_1\,\widehat{\sin}\,\hat{\theta} \qquad (111) \\
\hat{e}^{i(\hat{\phi}+\hat{\theta})} &= n_2\,\widehat{\cos}\,(\hat{\phi}+\hat{\theta}) + i\,n_1\,\widehat{\sin}\,(\hat{\phi}+\hat{\theta}) \qquad (112) \\
\hat{e}^{i(\hat{\phi}+\hat{\theta})} &= \hat{e}^{i\hat{\phi}} \times \hat{e}^{i\hat{\theta}} \,. \qquad (113)
\end{aligned}$$

2.3.8 Isotrigonometric Functions on an Isocircle

In order to gain a further insight into the *isotrigonometric functions* let us manipulate them on an *isocircle* of an *unit isoradius*.

In eqs.(82) and (88) the denominator consists of the lengths of the two *isovectors*. When both the *isovectors* are of the same length then their outer tips lie on an *isocircle* and their length measures the *isoradius* of that *isocircle*. However, when *isoradius* is unity then the denominator of these equations become unity in terms of the *isounit*. That get expressed as,

$$\begin{aligned}
\hat{r}_1^{\hat{2}} &= \left(x_1 n_1^{-2} x_1 + y_1 n_2^{-2} y_1\right) \times \hat{I}^{\hat{2}} = \hat{I}^{\hat{2}} \\
\hat{r}_2^{\hat{2}} &= \left(x_2 n_1^{-2} x_2 + y_2 n_2^{-2} y_2\right) \times \hat{I}^{\hat{2}} = \hat{I}^{\hat{2}} \qquad (114)
\end{aligned}$$

or in general we have,

$$\hat{r}^{\hat{2}} = \left(x\, n_1^{-2}\, x + y\, n_2^{-2}\, y\right) \times \hat{I}^{\hat{2}} = \hat{I}^{\hat{2}} \tag{115}$$

conforming to $\hat{r} = \hat{I}$ and hence we have,

$$x\, n_1^{-2}\, x + y\, n_2^{-2}\, y = 1 \tag{116}$$

that implies that,

$$\text{for} \quad y = 0 \quad \text{we have} \quad x = n_1, \quad \text{and for} \quad x = 0 \text{ we have } y = n_2 \tag{117}$$

that in terms of *isocoordinates* reads as,

$$\text{for} \quad \hat{y} = \hat{0} \quad \text{we have} \quad \hat{x} = n_1^3, \quad \text{and for} \quad \hat{x} = \hat{0} \text{ we have } \hat{y} = n_2^3 \tag{118}$$

But for $\hat{y} = 0$ we have $\hat{x} = \hat{r}$, therefore, we have $\hat{r} = n_1^3$ at $\hat{y} = 0$. Similarly, for $\hat{x} = \hat{0}$ we have $\hat{r} = n_2^3$. This then means that we can use n_1^3 and n_2^3 interchangeably whenever required.

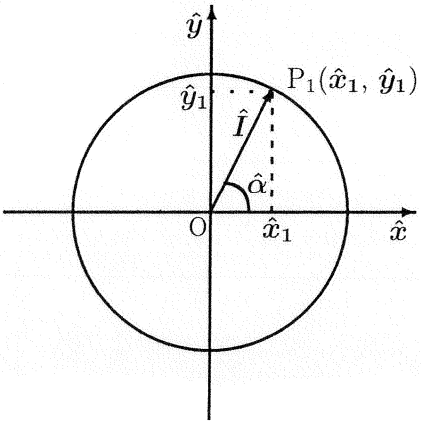


Figure 6: *Isounit-Isocircle*:

Now consider a pair of points $\mathrm{P}_1(\hat{x}_1, \hat{y}_1)$ and $\mathrm{P}_2(n_1^3, \hat{0})$ that correspond to $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(n_1, 0)$. Thus we see from Figure 6 that we are calculating *isocosine* of the *isoangle*, $\hat{\alpha}$, between the *isovencor* OP_1 and $\hat{x}$-axis. Therefore, from eq.(80) we have,

$$\cos\hat{\alpha} = x_1\, n_1^{-1} \tag{119}$$

Next we assume the pair of points $\mathrm{P}_1(\hat{x}_1,\ \hat{y}_1)$ and $\mathrm{P}_2(\hat{0},\ n_2^3)$ that correspond to $\mathrm{P}_1(x_1,\ y_1)$ and $\mathrm{P}_2(0,\ n_2)$. However, in this option we are calculating the *isocosine* of the *isoangle* formed between the chosen *isovector* and $\hat{y}$-axis. But from Figure 6 this *isoangle* is $\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\alpha}\right)$. Hence from eq.(80) we have,

$$\cos\left(\widehat{\left(\frac{\pi}{2}\right)} - \hat{\alpha}\right) = \sin\hat{\alpha} = n_2^{-1}\, y_1 \tag{120}$$

whereas the same result of eq.(120) is obtained directly from the expression of $\sin\hat{\alpha}$, eq.(85), on substituting $\mathrm{P}_2(n_1,\ 0)$.

In general the *isocosine* of an *isoangle* formed by an *isovector*, $\hat{r}$, and the $\hat{x}$-axis is given by,

$$\frac{\hat{x}}{\hat{r}} \times \hat{I} = \widehat{\cos}\hat{\alpha} \tag{121}$$

which gives,

$$\hat{x} = \hat{r}\hat{\times}\widehat{\cos}\hat{\alpha} \tag{122}$$

Similarly, for *isosine* we obtain,

$$\hat{y} = \hat{r}\hat{\times}\widehat{\sin}\hat{\alpha} \tag{123}$$

Now consider an *isocircle* of *isounit radius*, that is $\hat{r} = \hat{I}$ (c.f. Figure 6), then eqs.(122) and (123) yield,

$$\hat{x} = \hat{r}\hat{\times}\widehat{\cos}\hat{\alpha} = \hat{I} \times \hat{T} \times \widehat{\cos}\hat{\alpha} = \widehat{\cos}\hat{\alpha} \tag{124}$$

and

$$\hat{y} = \hat{r}\hat{\times}\widehat{\sin}\hat{\alpha} = \hat{I} \times \hat{T} \times \widehat{\sin}\hat{\alpha} = \widehat{\sin}\hat{\alpha} \tag{125}$$

But we have seen above that with reference to an *isocircle* of *isounit radius* we have (cf. eqs.(119) and (120)) in general,

$$x = n_1 \cos\hat{\alpha} \tag{126}$$

and

$$y = n_2 \sin\hat{\alpha} \tag{127}$$

From above discussion the isotopic lifting reads as,

$$\begin{aligned} x = \cos\alpha \quad &\longrightarrow \quad \hat{x} = \widehat{\cos}\hat{\alpha} \\ y = \sin\alpha \quad &\longrightarrow \quad \hat{y} = \widehat{\sin}\hat{\alpha} \end{aligned} \tag{128}$$

However, on projecting the above isofunction back to the original Euclidean plane produces,

$$\begin{aligned} \hat{x} = \widehat{\cos}\hat{\alpha} &\longrightarrow x = n_1 \cos\hat{\alpha} \\ \hat{y} = \widehat{\sin}\hat{\alpha} &\longrightarrow y = n_2 \sin\hat{\alpha} \end{aligned} \tag{129}$$

Now notice that numerically $\cos\alpha = \widehat{\cos}\hat{\alpha}$ and $\sin\alpha = \widehat{\sin}\hat{\alpha}$ but $\widehat{\cos}\alpha \neq \cos\hat{\alpha}$ and $\widehat{\sin}\alpha \neq \sin\hat{\alpha}$. This then reveals that we identically have,

$$\begin{aligned} x &= \widehat{\cos}\hat{\alpha} = n_1 \cos\hat{\alpha} \\ y &= \widehat{\sin}\hat{\alpha} = n_2 \sin\hat{\alpha} \end{aligned} \tag{130}$$

Thus the basic property of *isotrigonometry* in context with an *isocircle of unit isoradius* gets described as,

$$\begin{aligned} \hat{I} &= \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} \\ &= \widehat{\cos}^{\hat{2}}\hat{\alpha} + \widehat{\sin}^{\hat{2}}\hat{\alpha} \\ &= n_1^{-2}\widehat{\cos}^2\hat{\alpha} + n_2^{-2}\widehat{\sin}^2\hat{\alpha} \\ &= \cos^2\hat{\alpha} + \sin^2\hat{\alpha} \end{aligned} \tag{131}$$

or equivalently,

$$\begin{aligned} \hat{I} = \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} &= \widehat{\cos}^{\hat{2}}\hat{\alpha} + \widehat{\sin}^{\hat{2}}\hat{\alpha} \longrightarrow \\ \widehat{\cos}^2\hat{\alpha} + \widehat{\sin}^2\hat{\alpha} &= 1 \\ n_1^2\cos^2\hat{\alpha} + n_2^2\sin^2\hat{\alpha} &= 1 \end{aligned} \tag{132}$$

2.3.9 Invariancy of Geometry on Isotopic Lifting

Let us consider the case of a sphere to demonstrate that on isotopic lifting to *isosphere* preserves the geometry as well as the size [1]. Consider a sphere with unit radius, thus, with all identical semiaxes,

$$a = b = c = 1. \tag{133}$$

Now consider its deformation into ellipsoid with semiaxes in Euclidean space given by,

$$a = n_1^{-2},\ b = n_2^{-2},\ c = n_3^{-2} \tag{134}$$

But the basic unit in isospace is changed into the form $\hat{I} = \text{diag.}\,(n_1^2,\, n_2^2,\, n_3^2)$. Hence, we have,

$$\hat{a} = n_1^{-2} n_1^2 = 1,\ \hat{b} = n_2^{-2} n_2^2 = 1,\ \hat{c} = n_3^{-2} n_3^2 = 1 \tag{135}$$

Thus we see that the semiaxes in isospere preserve the original numerical value 1. This is one of the demonstrations of preservation of numerical value of $f(r)$ on isotopic lifting to $\hat{f}(\hat{r})$.

Moreover, this example is a manifestation of the following fundamental property of Euclidean and isoeuclidean geometries, namely;

$$(\text{Length})^2 \times (\text{unit})^2 = (\text{Isolength})^2 \times (\text{Isounit}^2) = \text{Invariant} \tag{136}$$

2.3.10 Isohyperbolic Functions

Recall that the hyperbolic cosine and sine functions are defined as,

$$\begin{aligned} \cosh\alpha &= \frac{e^{\alpha} + e^{-\alpha}}{2} \\ \sinh\alpha &= \frac{e^{\alpha} - e^{-\alpha}}{2} \end{aligned} \tag{137}$$

Hence we have,

$$\cosh\alpha + \sinh\alpha = e^{\alpha} \tag{138}$$

and the fundamental property of hyperbolic functions follows that reads as,

$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{139}$$

The *isotopically lifted* versions of the above expressions generate the *isohyperbolic functions* and corresponding identities. That read as,

$$\begin{aligned} \widehat{\cosh}\hat{\alpha} &= \frac{\hat{e}^{\alpha} + \hat{e}^{-\alpha}}{2} = \hat{I}_{\hat{\alpha}} \times \frac{e^{\alpha \times \hat{T}_{\hat{\alpha}}} + e^{-\alpha \times \hat{T}_{\hat{\alpha}}}}{2} = \hat{I}_{\hat{\alpha}} \times \cosh\hat{\alpha} \\ \widehat{\sinh}\hat{\alpha} &= \frac{\hat{e}^{\alpha} - \hat{e}^{-\alpha}}{2} = \hat{I}_{\hat{\alpha}} \times \frac{e^{\alpha \times \hat{T}_{\hat{\alpha}}} - e^{-\alpha \times \hat{T}_{\hat{\alpha}}}}{2} = \hat{I}_{\hat{\alpha}} \times \sinh\hat{\alpha}, \end{aligned} \tag{140}$$

$$\widehat{\cosh}^{\hat{2}}\hat{\alpha} - \widehat{\sinh}^{\hat{2}}\hat{\alpha} = \hat{I}_{\hat{\alpha}} \longrightarrow n_1^2\cosh^2\hat{\alpha} - n_2^2\sinh^2\hat{\alpha} = 1 \tag{141}$$

and

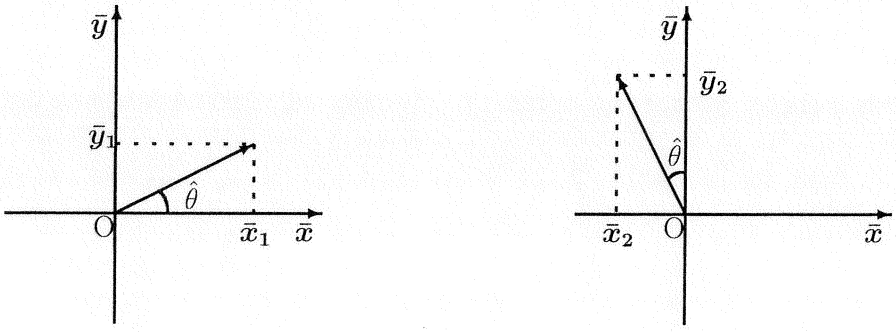
$$\hat{e}^{\alpha} = \hat{I}_{\hat{\alpha}} \times e^{\alpha \times \hat{T}_{\hat{\alpha}}} = \widehat{\cosh}\hat{\alpha} + \widehat{\sinh}\hat{\alpha} = n_1\cosh\hat{\alpha} + n_2\sinh\hat{\alpha} \tag{142}$$

2.3.11 Isorotation Symmetry

Let us first consider an *unit isovector* in (x, y) flat plane, which from eq.(116) using the definitions of eq.(65) reads as,

$$\bar{x} \times \bar{x} + \bar{y} \times \bar{y} = 1 \tag{143}$$

This vector when lies on $\bar{x}$-axis it gets described by (1, 0). When this vector is rotated by an *isoangle*, $\hat{\theta}$ anticlocwise in the plane $(\bar{x}, \bar{y})$ the resultant isovector gets described by say $(\bar{x}_1, \bar{y}_1)$. It is shown in Figure 7(a). We have



(a) (b)

Figure 7: *Rotation of an Unit-Isovector*: Figure (a) shows the counterclockwise rotated unit isovector (1,0) by an isoangle $\hat{\theta}$; and Figure 7(b) depicts the counterclockwise rotated unit isovector (0,1) by an isoangle $\hat{\theta}$

depicted the new coordinates by $(\bar{x}_1, \bar{y}_1)$. These coordinates get quantified as,

$$\bar{x}_1 = \cos\hat{\theta}, \qquad \bar{y}_1 = \sin\hat{\theta} \tag{144}$$

Similarly, the unit isovector (0,1) when rotated counterclockwise by an isoangle $\hat{\theta}$ the new isovector is $(\bar{x}_2, \bar{y}_2)$ (cf. Figure 7(b)). The new coordinates are quantified as,

$$\bar{x}_2 = -\sin\hat{\theta}, \qquad \bar{y}_2 = \cos\hat{\theta} \tag{145}$$

Therefore, the *isorotation* matrix is given by,

$$\begin{pmatrix} \cos\hat{\theta} & -\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix} \tag{146}$$

Now any isovector $(\bar{x}, \bar{y})$ when rotated by an isoangle say $\hat{\theta}$ the new coordinates would be say $(\bar{x}', \bar{y}')$. This transformation gets represented by,

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} \cos\hat{\theta} & -\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \tag{147}$$

Therefore, the equations of the transformations of coordinates in $\bar{E}(\bar{r}, \delta, R)$ plane read as,

$$\bar{x}' = \bar{x}\cos\hat{\theta} - \bar{y}\sin\hat{\theta} \tag{148}$$

$$\bar{y}' = \bar{x}\sin\hat{\theta} + \bar{y}\cos\hat{\theta} \tag{149}$$

Now, the substitution of the definitions of eq.(65) in the preceding equations and on rearrangement gives,

$$x' = x\cos\hat{\theta} - \frac{n_1}{n_2} y\sin\hat{\theta} \tag{150}$$

$$y' = \frac{n_2}{n_1} x\sin\hat{\theta} + y\cos\hat{\theta} \tag{151}$$

that in matrix notation reads as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\hat{\theta} & -\frac{n_1}{n_2}\sin\hat{\theta} \\ \frac{n_2}{n_1}\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{152}$$

and on using the azimuthal angular transformation depicted in eq.(92) we explicitly have,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\left(n_1^{-1}n_2^{-1}\theta\right) & -\frac{n_1}{n_2}\sin\left(n_1^{-1}n_2^{-1}\theta\right) \\ \frac{n_2}{n_1}\sin\left(n_1^{-1}n_2^{-1}\theta\right) & \cos\left(n_1^{-1}n_2^{-1}\theta\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{153}$$

For the more abstract derivation of above transformation reader is advised to refer [1, 11, 22, 23].

2.4 Operations on Isovectors

Let us consider the operations of isovectors in the *isoeuclidean* geometry using diagonal isounit $\hat{I} == \widehat{T}^{-1} = \text{diag.}\,(n_1^2,\, n_2^2,\, n_3^2)$. Let e_k, $k = 1, 2, 3$ be

the unit vectors of a three-dimensional Euclidean space $E(r,\ \delta,\ R)$ directed along the x, y and z axes, and let,

$$\hat{e}_k = e_k \times \hat{I} \tag{154}$$

be the corresponding *isobasis* in $\hat{E}(\hat{r},\ \hat{\delta},\ \hat{R})$. Then the *isovector*, $\hat{V}$ gets expressed in isospace as,

$$\hat{V} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 \tag{155}$$

Notice that this is an another way of expressing the fact that the isovector $\hat{V}$ is isostraight in $\hat{E}(\hat{r},\ \hat{\delta},\ \hat{R})$, although its projection in $E(r,\ \delta,\ R)$ is curved.

Recall that the equation of a straight line passing through a given point say P_1 with position vector $\mathbf{r}_1$ and parallel to a unit vector a is given by,

$$\mathbf{r} = \mathbf{r}_1 + n\mathbf{a} \tag{156}$$

whose components are described as,

$$r_k - r_{1k} - na_k = 0, \quad k = 1, 2, 3 \tag{157}$$

where n is some number. Now the isotopic lifting of above equation gives the equation of an *isoline* in *isovector* notation that reads as,

$$\hat{r}_k - \hat{r}_{1k} - \hat{n}\hat{\times}\hat{a}_k = (r_k - r_{1k} - na_k) \times \hat{I} = 0, \quad k = 1, 2, 3 \tag{158}$$

As expected, *the operations on vectors are preserved under isotopies.* In fact, the familiar scalar product of two vectors $V_1 = (x_1,\ y_1,\ z_1)$ and $V_2 = (x_2,\ y_2,\ z_2)$

$$V_1 \odot V_2 = x_1x_2 + y_1y_2 + z_1z_2 \tag{159}$$

is now lifted into the following expression called *isoscalar product*,

$$\hat{V}_1\hat{\odot}\hat{V}_2 = \left(x_1n_1^{-2}x_2 + y_1n_2^{-2}y_2 + z_1n_3^{-2}z_2\right) \times \hat{I} \tag{160}$$

Notice that the *isoscalar product* preserves the original axioms, namely:

$$\hat{V}_1\hat{\odot}\hat{V}_2 = \hat{V}_2\hat{\odot}\hat{V}_1,\ \hat{V}_1\hat{\odot}(\hat{V}_2 + \hat{V}_3) = \hat{V}_1\hat{\odot}\hat{V}_2 + \hat{V}_1\hat{\odot}\hat{V}_3 \tag{161}$$

Moreover, the *isonorm* on $\hat{E}(\hat{r},\ \hat{\delta},\ \hat{R})$ is expressible in terms of the isoscalar product via the rule,

$$\hat{|}\hat{V}\hat{|} = (\hat{V}\hat{\odot}\hat{V})^{\frac{1}{2}} \times \hat{I} \tag{162}$$

Thus the *isocosine* of an isoangle formed by two intersecting isovectors can be written as the isotopy of the conventional case,

$$\widehat{\cos}\hat{\alpha} = \frac{\hat{V}_1 \hat{\odot} \hat{V}_2}{\hat{|}\hat{V}_1\hat{|}\hat{\times}\hat{|}\hat{V}_2\hat{|}} \tag{163}$$

Now let $\hat{V}$ be an isovector and $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ are its components along three isoeuclidean axes. Further, let the isovector $\hat{V}$ makes angles $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ with respective axes. Then the *directional isocosines* are given by,

$$\widehat{\cos}\hat{\alpha} = \frac{\hat{V}_1}{\hat{|}\hat{V}\hat{|}}, \quad \widehat{\cos}\hat{\beta} = \frac{\hat{V}_2}{\hat{|}\hat{V}\hat{|}}, \quad \widehat{\cos}\hat{\gamma} = \frac{\hat{V}_3}{\hat{|}\hat{V}\hat{|}} \tag{164}$$

This then produces the correct isotopic lifting of the corresponding conventional identity to:

$$n_1^{-2}\widehat{\cos}^2\hat{\alpha} + n_2^{-2}\widehat{\cos}^2\hat{\beta} + n_3^{-2}\widehat{\cos}^2\hat{\gamma} = 1 \tag{165}$$

Similarly, the vectorial product $V_1 \wedge V_2$ gets isotopically lifted to *isovectorial product*, that reads as,

$$\hat{V}_3 = \hat{V}_1 \hat{\wedge} \hat{V}_2, \;\; V_{3k} = \hat{\epsilon}_{kij}(n_i^{-1}x_{i1})(n_j^{-1}x_{2j}), \;\; i,j,k = 1,2,3 \tag{166}$$

that satisfies the basic axioms of the vectorial products, viz.,

$$\hat{V}_1 \hat{\wedge} \hat{V}_2 = -\hat{V}_2 \hat{\wedge} \hat{V}_1, \;\; \hat{V}_1 \hat{\wedge} (\hat{V}_2 + \hat{V}_3) = \hat{V}_1 \hat{\wedge} \hat{V}_2 + \hat{V}_1 \hat{\wedge} \hat{V}_3 \tag{167}$$

and the *Lagrange's identity* under isotopies among the four isovectors $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ in $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ follows,

$$(\hat{A}\hat{\wedge}\hat{B})\hat{\odot}(\hat{C}\hat{\wedge}\hat{D}) = (\hat{A}\hat{\odot}\hat{C})\hat{\times}(\hat{B}\hat{\odot}\hat{D}) - (\hat{B}\hat{\odot}\hat{C})\hat{\times}(\hat{A}\hat{\odot}\hat{D}) \tag{168}$$

Other properties can be derived too.

2.5 Isodifferentiation and Isointegration

The isotopic lifting of the operation of differentiation is required to be handled in such a way that on operating the same isounit is maintained. The

convention adopted by Santilli in reference [9] has been used herein. The *isodifferential* is instructively generated as follows,

$$dr = \widehat{T} \times dr \times \hat{I} \xrightarrow{\hat{I}=const.} \widehat{T} \times d(r \times \hat{I}) = \hat{d}\hat{r} \tag{169}$$

Now assume $\hat{I} \neq constant$ that leads us to,

$$\hat{d}\hat{r} = \widehat{T} \times d\hat{r} = \widehat{T} \times d(r \times \hat{I}) \neq dr \tag{170}$$

Notice that the isotopy of differentiation doesn't get generated by nonunitary or noncanonical transform.

The *ordinary derivative* of an isofunction is evidently given by,

$$\frac{\partial \hat{f}(\hat{r}_i)}{\partial \hat{r}_i} \stackrel{\text{def.}}{\equiv} \lim_{\triangle \hat{r}_i \to 0} \frac{\hat{f}(\hat{r}_i + \triangle \hat{r}_i) - \hat{f}(\hat{r}_i)}{\triangle \hat{r}_i} \tag{171}$$

Thus the *isoderivative* of *isofunctions* then follows using *isocoordinates* as,

$$\frac{\hat{\partial} \hat{f}(\hat{r}_i)}{\hat{\partial} \hat{r}_i} = \hat{I}_{ji} \times \frac{\partial \hat{f}(\hat{r}_j)}{\partial \hat{r}_i} \tag{172}$$

The left hand side of the preceding equation is an *isoquotient* and the additional *isounit* has been absorbed in $\hat{f}$ [9].

Now we consider a few simple examples to illustrate the axiom-preserving character of the *isodifferential calculus*. In all the following examples, for the sake of simplicity, we assume that the *isounit* is not dependent on $\hat{r}$.

1. Consider the case of $\hat{f}(\hat{r}_i) = \hat{r}_i$, then we have,

$$\frac{\hat{d}\hat{r}_i}{\hat{d}\hat{r}_j} = \hat{\delta}_{ij} = \delta_{ij} \times \hat{I} \tag{173}$$

and similarly we have,

$$\frac{\hat{d}(\hat{r}_i)^{\hat{n}}}{\hat{d}\hat{r}_j} = \hat{n} \times \hat{\delta}_{ij} \times (\hat{r}_i)^{\hat{n}-\hat{1}} = \hat{\delta}_{ij} \times (\hat{r}_i)^{\hat{n}-\hat{1}} \tag{174}$$

2. The *isolaplacian* reads as,

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}_i}\frac{\hat{\partial}}{\hat{\partial}\hat{r}_j} = \frac{\hat{\partial}}{\hat{\partial}\hat{r}_i} \times \hat{\delta}_{ij} \times \frac{\hat{\partial}}{\hat{\partial}\hat{r}_j} = \frac{\hat{\partial}}{\hat{\partial}\hat{r}_i} \times \delta_{ij} \times \widehat{T}_{ij} \times \frac{\hat{\partial}}{\hat{\partial}\hat{r}_j} \tag{175}$$

3. The *isoderivatives in different variables "isocommute" on isospace over isofields,*

$$\frac{\hat{\partial}}{\partial \hat{r}_i}\frac{\hat{\partial}}{\partial \hat{r}_j} = \frac{\hat{\partial}}{\partial \hat{r}_j}\frac{\hat{\partial}}{\partial \hat{r}_i} \tag{176}$$

but their projections on the ordinary spaces over ordinary fields do not necessarily "commute".

4. The axiom-preserving character of isoderivative of *isoexponent* gets demonstrated as follows,

$$\frac{\hat{\partial}}{\partial \hat{r}}\,\hat{e}^{\hat{r}} = \hat{I} \times \frac{\partial}{\partial \hat{r}}\left[\hat{I} \times e^{\widehat{T}\times\hat{r}}\right] = \hat{I} \times \widehat{T} \times \left[\hat{I} \times e^{\widehat{T}\times\hat{r}}\right] = \hat{e}^{\hat{r}} \tag{177}$$

5. Consider now the *isoplanewave* as a simple isotopy of the conventional planewave solution,

$$\hat{e}^{\hat{i}\hat{\times}\hat{K}\hat{\times}\hat{r}} = \hat{I} \times e^{i\times\widehat{T}\times K\times\hat{r}}, \tag{178}$$

for which we have the *isoderivatives,*

$$\begin{aligned}\frac{\hat{\partial}}{\partial \hat{r}}\,\hat{e}^{\hat{i}\hat{\times}\hat{K}\hat{\times}\hat{r}} &= \hat{I} \times \frac{\partial}{\partial \hat{r}}\left[\hat{I} \times e^{i\times\widehat{T}\times K\times\hat{r}}\right] \\ &= i \times K \times \hat{I} \times e^{i\times\widehat{T}\times K\times\hat{r}} = \hat{i}\hat{\times}\hat{K}\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{K}\hat{\times}\hat{r}}\end{aligned} \tag{179}$$

6. From the above demonstration we arrive at the following fundamental definition of *isomomentum* first time realized by Santilli in references [1,11],

$$\hat{p}_k\hat{\times}\hat{\psi}(\hat{t},\,\hat{r}) = -\hat{i}\hat{\times}\frac{\hat{\partial}}{\partial \hat{r}_k}\hat{\psi}(\hat{t},\,\hat{r}) = -\hat{i}\hat{\times}\hat{I}_{ki} \times \frac{\partial}{\partial \hat{r}_k}\hat{\psi}(\hat{t},\,\hat{r}) = \hat{K}\hat{\times}\hat{\psi}(\hat{t},\,\hat{r}) \tag{180}$$

That is the quantum mechanical momentum equation got isotopically lifted as,

$$p_k \times \psi(t,\,r) = -i \times \hbar \times \frac{\partial}{\partial r}\psi(t,\,r) \equiv -i \times \frac{\hat{\partial}}{\hat{\partial} r}\psi(t,\,r) \tag{181}$$

Notice that the quantum mechanical eigenvalue equations get identically reformulated in terms of the isodifferential calculus with basic isounit $\hbar$.

Now we quote below from the reference [9].

In conclusion, Santilli's isodifferential calculus establishes that the isounit is not only an algebraic unit of hadronic mechanics but also replaces Planck's constant with an integro-differential operator $\hat{I}$, as needed to represent contact, nonlinear, nonlocal and nonpotential effects.

More specifically, Santilli's isodifferential calculus establishes that, while in exterior dynamical systems such as atomic structure, we have the conventional quantization of energy, while in interior dynamical systems such as in the structure of hadrons, nuclei and stars we have superposition of quantized energy levels at atomic distances plus continuous energy exchanges at hadronic distances. This we have shown in Figure 8 for atomic orbitals of a Hydrogen atom and neutron as deep overlapping of the wave packets of proton and electron on collapsing of Hydrogen atom.

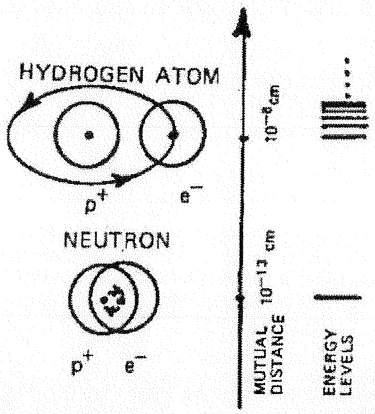


Figure 8: Santilli's hadronic mechanics has identified the existence of an additional bound state when the electron orbits within the proton structure at distances of the order of 10^{-13} cm or less. Remarkably, Santilli has proved that the hadronic state is one and one only, the neutron, because, when excited, the electron leaves the proton structure, thus recovering all conventional quantum states. In this sense, the energy levels of the hydrogen atom are the excited states of the neutron.

Needless to say, that all models of hadronic mechanics will be restricted by the condition,

$$\lim_{r\to\infty} \hat{I} \equiv \hbar \tag{182}$$

under which hadronic mechanics recovers quantum mechanics uniquely and identically.

Correspondingly, the *isointegration* in the case of the constancy of isounit gets defined as follows,

$$\hat{\int} = \hat{I} \times \int \tag{183}$$

and hence we have,

$$\hat{\int} \hat{d}\hat{r} = \hat{I} \times \int \widehat{T} \times d(r \times \hat{I}) = \hat{I} \times \int dr \tag{184}$$

The *time isodifferentials* and *isoderivatives* are defined accordingly via the use of the scalar *isounit of time*, $\hat{I}_{\hat{t}} = 1/\widehat{T}_{\hat{t}}$,

$$\hat{d}\hat{t} = \hat{I}_{\hat{t}} \times d\hat{t}, \quad \frac{\hat{\partial}}{\hat{\partial}\hat{t}} = \widehat{T}_{\hat{t}} \times \frac{\partial}{\partial t} \tag{185}$$

Indeed, the above presentation is that of contravariant version of the isodifferential calculus which is the counterpart of the normal differential calculus. The covariant version may be discussed on similar lines but we have refrained ourselves for the sake of brevity. The interested reader can refer to the original literature cited in this paper.

2.6 Newton-Santilli Isomechanics

With the above described rudiments of isomathematics we are now in a position to isotopically lift the Newtonian mechanics. Santilli lucidly described this isotopiclifting as follows (see for example Refs. [9,24]).

2.6.1 Conventional Newtonian Mechanics

Let us represent the Newtonian mechanics in its $7N$ - dimensional space (where N is the total number of particles) [9,24],

$$m_a \times \frac{dv_{ak}}{dt} = F_{ak}(t,\, r\, v), \quad k = 1,2,3, \quad a = 1,2,3,.....,N \tag{186}$$

where F_{ak} is the k^{th} component of the body force on the particle a, m_a is the mass of particle a and v_{ak} is the k^{th} component of the velocity of the particle a.

The $7N$ - dimensional Newtonian space is determined by the Kronecker product of the Euclidean spaces of time t, coordinates r and velocities v as,

$$S(t,\, r,\, v) = E(t,\, R_t) \times E(r,\, \delta,\, R_r) \times E(v,\, \delta,\, R_v) \tag{187}$$

with

$$r = (r_{ak}) = (r_{a1},\, r_{a2},\, r_{a3}) = (x_a,\, y_a,\, z_a) \tag{188}$$

$$v = (v_{ak}) = (v_{a1},\, v_{a2},\, v_{a3}) = \frac{dr}{dt} \tag{189}$$

$$\delta = \text{Diag.}(1,\, 1,\, 1), \quad k = 1, 2, 3, \quad a = 1, 2, 3, ..., N \tag{190}$$

and the base fields are trivially identical, i.e., $R_t = R_r = R_v$, since all units are assumed to have the trivial value +1, resulting in the trivial total unit,

$$I_{tot} = I_t \times I_r \times I_v = 1 \times 1 \times 1 = 1. \tag{191}$$

where R_t, R_r and R_v are are the time, coordinates and velocity fields respectively that generate through Kronecker product the total field R_{tot} as,

$$R_{tot} = R_t \times R_r \times R_v \tag{192}$$

and the unit of time is the scalar $I_t = 1_t$, while the unit of the Euclidean space is the unit matrix and the same happens for the velocities $I_r = I_v = \text{Diag.}(1,\, 1,\, 1)$. Hence eq.(191) reads as,

$$I_{tot} = 1_t \times \text{Diag.}(1,\, 1,\, 1) \times \text{Diag.}(1,\, 1,\, 1) \tag{193}$$

The above total unit can be factorized into the product of seven individual units for time and two sets of individual Euclidean axes x, y and z with corresponding factorization of fields,

$$I_{tot} = 1_t \times 1_{rx} \times 1_{ry} \times 1_{rz} \times 1_{vx} \times 1_{vx} \times 1_{vx} \tag{194}$$

$$R_{tot} = R_t \times R_{rx} \times R_{ry} \times R_{rz} \times R_{vx} \times R_{vx} \times R_{vx} \tag{195}$$

This constitutes the foundations of the conventional *Euclidean topology*.

In the Newtonian equation (186) the force term can be generalized to include non-body forces. The latter ones are the contact forces, such as,

frictional forces originating from the contact of the given body with a viscous medium through which it moves [25]. It has been shown by Santilli [26, 27] that the right hand side of eq.(186) on including all types of forces gets categorized into *selfadjoint forces*, F_{ak}^{SA}, and *non-selfadjoint forces*, F_{ak}^{NSA}. In the view of this eq.(186) gets expressed as,

$$m_a \times \frac{dv_{ak}}{dt} = F_{ak}(t,\, r\, v) = F_{ak}^{SA} + F_{ak}^{NSA} \tag{196}$$

with by definition,

$$\sum_a \mathbf{F}_a^{NSA} = 0, \quad \sum_a \mathbf{r}_a \odot \mathbf{F}_a^{NSA} = 0, \quad \sum_a \mathbf{r}_a \wedge \mathbf{F}_a^{NSA} = 0 \tag{197}$$

2.6.2 Newoton-Santilli Isomechanics

The isotopies of Newtonian mechanics, now a days is known as *Newton-Santilli isomechanics* (see for example Ref. [9]). It requires the use of *isotime* $\hat{t} = t \times \hat{I}_t$ with corresponding *isounit* $\hat{I}_t = 1/\widehat{T}_t > 0$ and related *isofield* $\hat{R}_t$; the *isocoordinates* $\hat{r} = (\hat{r}_{ak}) = r \times \hat{I}_r$, with its *isounit* $\hat{I}_r = 1/\widehat{T}_r > 0$ and related *isofield* $\hat{R}_r$; and the *isovelocities* $\hat{v} = (\hat{v}_{ak}) = v \times \hat{I}_v$ with its *isounit* $\hat{I}_v = 1/\widehat{T}_v > 0$ and related *isofield* $\hat{R}_v$.

The *Newton-Santilli isomechanics* is then formulated on the 7-*dimensional isospace* as,

$$\hat{S}(\hat{t},\, \hat{r},\, \hat{v}) = \hat{E}(\hat{t},\, \hat{R}_{\hat{t}}) \times \hat{E}(\hat{r},\, \hat{\delta},\, \hat{R}_{\hat{r}}) \times \hat{E}(\hat{v},\, \hat{\delta},\, \hat{R}_{\hat{v}}) \tag{198}$$

with corresponding *isomatrics*,

$$\hat{\delta}_{\hat{r}} = \widehat{T}_{\hat{r}} \times \delta = \left(\widehat{T}_{\hat{r}i}^{k} \times \delta_{kj}\right), \quad \hat{\delta}_{\hat{v}} = \widehat{T}_{\hat{v}} \times \delta = \left(\widehat{T}_{\hat{v}i}^{k} \times \delta_{kj}\right) \tag{199}$$

over the Kronecker product of isofields,

$$\hat{R}_{tot} = \hat{R}_{\hat{t}} \times \hat{R}_{\hat{r}} \times \hat{R}_{\hat{v}} \tag{200}$$

with total *isounit*,

$$\begin{aligned} &\hat{I}_{tot} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \hat{I}_{\hat{v}} = \\ = \;& n_t^2 \times \text{Diag.}\left(n_{rx}^2,\, n_{ry}^2,\, n_{rz}^2\right) \times \text{Diag.}\left(n_{vx}^2,\, n_{vy}^2,\, n_{vz}^2\right). \end{aligned} \tag{201}$$

Consequently, the isounit can also be factorized into the product of the following seven distinct isounits, with related product of seven distinct isofields,

$$\hat{I}_{tot} = n_t^2 \times n_{rx}^2 \times n_{ry}^2 \times n_{rz}^2 \times n_{vx}^2 \times n_{vy}^2 \times n_{vz}^2 \tag{202}$$

$$\hat{R}_{tot} = \hat{R}_t^2 \times \hat{R}_{rx}^2 \times \hat{R}_{ry}^2 \times \hat{R}_{rz}^2 \times \hat{R}_{vx}^2 \times \hat{R}_{vy}^2 \times \hat{R}_{vz}^2, \tag{203}$$

which for the first time allows a consistent representation of extended, nonsherical and deformable shapes of particles in Newtonian mechanics, here represented via the semiaxes $n_\alpha^2 = n_\alpha^2(t,\, r\, v,\, ...),\ \alpha = t,\, r,\, v$.

Note that the *isovelocity* is given by,

$$\hat{v} = \frac{\hat{d}\hat{r}}{\hat{d}\hat{t}} = \hat{I}_{\hat{t}} \times \frac{d\left(r \times \hat{I}_{\hat{r}}\right)}{dt} = v \times \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} + r \times \hat{I}_{\hat{t}} \times \frac{d\hat{I}_{\hat{r}}}{dt} = v \times \hat{I}_{\hat{v}}, \tag{204}$$

thus illustrating that the isounit of the isovelocity cannot be the same as that for the isocoordinates, having its value,

$$\hat{I}_{\hat{v}} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \left(1 + \frac{r}{v} \times \frac{1}{\hat{I}_{\hat{r}}} \times \frac{d\hat{I}_{\hat{r}}}{dt}\right) \tag{205}$$

Then the *Newton-Santilli isoequation* gets expressed as,

$$\hat{m}_a \hat{\times} \frac{\hat{d}\hat{v}_{ak}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}_{ak}}, \tag{206}$$

namely, *the equations are conceived in such a way to formally coincide with the conventional equations for selfadjoint forces when formulated on isospace over isofields, while all nonpotential forces are represented by the isounits or, equivalently, by the isodifferential calculus.*

An inspection of eqs.(204) - (206) reveals that the *Newton - Santilli isomechanics reconstructs linearity, locality and canonicity on isospaces over isofields.* However, this would not be the case if *nonselfadjoint* forces appear on the right hand side of eq.(206) as in eq.(196).

When projected in the conventional representation space S_{tot}, eq.(206)

for a given particle gets explicitly expressed as,

$$\begin{aligned} \hat{m}\hat{\times}\frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} &= m \times \hat{I}_{\hat{t}} \times \frac{d(v \times \hat{I}_{\hat{v}})}{dt} \\ &= m \times \frac{dv}{dt} \times \hat{I}_{\hat{t}} \times \hat{I}_{\hat{v}} + m \times v \times \hat{I}_{\hat{t}} \times \frac{d\hat{I}_{\hat{v}}}{dt} \\ &= -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}} = -\hat{I}_{\hat{r}} \times \frac{\partial V}{\partial r}, \end{aligned} \tag{207}$$

that is

$$m \times \frac{dv}{dt} = -\widehat{T}_{\hat{t}} \times \widehat{T}_{\hat{v}} \times \hat{I}_{\hat{r}} \times \frac{\partial V}{\partial r} - m \times v \times \widehat{T}_{\hat{v}} \times \frac{d\hat{I}_{\hat{v}}}{dt} \tag{208}$$

The necessary and sufficient conditions for the representation of all possible SA and NSA forces are given by,

$$\hat{I}_{\hat{r}} = \widehat{T}_{\hat{t}} \times \widehat{T}_{\hat{r}}, \quad m \times v \times \widehat{T}_{\hat{v}} \times \frac{d\hat{I}_{\hat{v}}}{dt} = F^{NSA} \tag{209}$$

As a simple illustration among unlimited possibilities, we have the following equations of motion of an *extended* particle with the ellipsoidal shape experiencing a resistive force $F^{NSA} = -\gamma \times v$ because of moving within a physical medium,

$$m \times \frac{dv}{dt} = \int d\sigma \Gamma(\sigma, \mathbf{r}, p, ...) \approx -\gamma \times v, \quad \hat{I}_{\hat{v}} = \text{Diag.}\left(n_1^2, n_2^2, n_3^2\right) \times e^{\gamma \times v}, \tag{210}$$

where the nonlocal-integral character with respect to a kernel Γ is emphasized.

2.7 Iso-Schrödinger Equation and Some of the Related Aspects

The isotopic lifting of Schrödinger wave equation is achieved in the following way:

$$
\begin{aligned}
H\times \mid \psi\rangle = E^0 \mid \psi\rangle \quad \longrightarrow \quad & U \times (H\times \mid \psi\rangle) \\
& = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \\
& \quad \times (U\times \mid \psi\rangle) \\
& = \hat{H} \times \hat{T}\times \mid \hat{\psi}\rangle = \hat{H}\hat{\times} \mid \hat{\psi}\rangle \\
& = U \times (E\times \mid \psi\rangle) \\
& = (U \times E \times U^\dagger) \times (U \times U^\dagger)^{-1} \\
& \quad \times (U\times \mid \psi\rangle) \\
& = \hat{E}\hat{\times} \mid \hat{\psi}\rangle = E \mid \hat{\psi}\rangle \qquad (211)
\end{aligned}
$$

That is we have arrived at the *iso-Schrödinger* equation, namely:

$$\hat{H}\hat{\times} \mid \hat{\psi}\rangle = E \mid \hat{\psi}\rangle \qquad (212)$$

Notice that the involved definition of ket *isowavefunction* is.

$$\mid \hat{\psi}\rangle = U\times \mid \psi\rangle \qquad (213)$$

and we have not lifted E^0 to its iso-counterpart because the new iso-Hamiltonian operator iso-operates on isowavefunction which is bound to produce different value of corresponding energy eigenvalue. Similarly, the bra *isowavefunction* gets generated as,

$$\langle\hat{\psi}\mid = (\langle\psi \mid \times U^\dagger) \qquad (214)$$

Notice that we have used $\hat{H}\hat{\times} \mid \hat{\psi}\rangle$, rather than $\hat{H}\times \mid \hat{\psi}\rangle$. In fact, only the former admits $\hat{I}$ as the correct isounit, that gets demonstrated as,

$$\hat{I}\hat{\times} \mid \hat{\psi}\rangle \; = \; \hat{T}^{-1} \times \hat{T}\times \mid \hat{\psi}\rangle \; \equiv \mid \hat{\psi}\rangle \qquad (215)$$

Some of the other relevant *isooperations* are:

1. The *isoinner product* is defined as,

$$
\begin{aligned}
\langle\psi \mid \times \mid \phi\rangle \quad \rightarrow \quad & U \times \langle\psi \mid \times \mid \phi\rangle \times U^\dagger \\
& = \left(\langle\psi \mid \times U^\dagger\right) \times \left(U \times U^\dagger\right)^{-1} \\
& \quad \times (U\times \mid \phi\rangle) \times \left(U \times U^\dagger\right) \\
& = \langle\hat{\psi}|\hat{\times}|\,\hat{\phi}\rangle \times \hat{I} \qquad (216)
\end{aligned}
$$

That has the following expression,

$$\langle\hat{\psi}|\hat{\times}|\,\hat{\phi}\rangle \times \hat{I} \;=\; \hat{I} \times \hat{\int} \hat{dr}^3\, \hat{\psi}^{\hat{\dagger}}(\hat{r}) \times \hat{T}(r, \cdots) \times \hat{\phi}(\hat{r}) \tag{217}$$

2. The *isonormalization* is defined as,

$$\langle\hat{\psi}^{\hat{\dagger}}|\,\hat{\psi}\rangle \;= \langle\hat{\psi}|\hat{\times}|\,\hat{\psi}\rangle = \;1 \tag{218}$$

3. The *isouncertainties* are defined in general as,

$$\Delta x\,\Delta k \geqq \frac{1}{2}\langle\hat{I}\rangle, \quad \Delta x \approx a/\hat{T}^{1/2},\; \Delta k \approx 1/(a\hat{T}^{1/2}),\; \lim_{r\to\infty} \hat{I} \equiv \hbar \tag{219}$$

4. The *isounitary operators* are the *isooperators*, namely:

$$\hat{U}\hat{\times}\hat{U}^{\hat{\dagger}} = \hat{U}^{\hat{\dagger}}\hat{\times}\hat{U} = \hat{I} \tag{220}$$

5. The notion of *isoexpectation value* of an *isooperator*, $\hat{H}$, is given by,

$$\langle\hat{H}\rangle \;=\; \frac{\langle\hat{\psi}|\hat{\times}\hat{H}\hat{\times}|\,\hat{\psi}\rangle}{\langle\hat{\psi}|\hat{\times}|\,\hat{\psi}\rangle} \tag{221}$$

Note also that the *isoexpectation value of the isounit is the isounit*,

$$\langle\hat{I}\rangle \;=\; \hat{I} \tag{222}$$

provided, of course, that one uses the isoquotient (otherwise $\langle\hat{I}\rangle \;= I$).

6. The time dependent Schrödinger equation is,

$$i \times \hbar\frac{\partial}{\partial t}\,|\,\psi\rangle = H\times\,|\,\psi\rangle = E\times\,|\,\psi\rangle \tag{223}$$

Its isotopic lifting reads as,

$$\begin{aligned} U \times (i \times \hbar\frac{\partial}{\partial t}\,|\,\psi\rangle) \;&=\; U \times (H\times\,|\,\psi\rangle) = U \times (E\times\,|\,\psi\rangle) \\ &=\; \hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\,|\,\hat{\psi}\rangle \;=\; \hat{H}\hat{\times}\,|\,\hat{\psi}\rangle \\ &=\; \hat{E}\hat{\times}\,|\,\hat{\psi}\rangle \;=\; E\times\,|\,\hat{\psi}\rangle \end{aligned} \tag{224}$$

where,

$$|\,\hat{\psi}(\hat{t})\rangle \;=\; \hat{U}\hat{\times}\,|\,\hat{\psi}(\hat{0})\rangle \;=\; \left\{\hat{e}^{\hat{i}\hat{\times}\hat{H}\hat{\times}\hat{t}}\right\}\hat{\times}\,|\,\hat{\psi}(\hat{0})\rangle \tag{225}$$

2.8 Some Examples of Isounits

An example of an *isounit* used above is the diagonal matrix of *isoeuclidean space*, namely:

$$\hat{I} = Diag.(n_1^2,\ n_2^2,\ n_1^2)) \tag{226}$$

Another example that represents interactions due to wavepacket-overlapings is,

$$\hat{I} = e^{N(\psi)\times\int dx\times\psi^\dagger(x)\psi(x)} \tag{227}$$

that in the case of two spinning particles with point like charges (such as electrons) in the case of deep penetration of their wave packets is given by,

$$\hat{I} = e^{\Gamma(\psi,\,\psi^\dagger)\times\int dv\times\psi_\downarrow^\dagger(r)\psi_\uparrow(r)} \tag{228}$$

where the nonlinearity is expressed by $\Gamma(\psi,\ \psi^\dagger)$ and the nonlocality is expressed by the volume integral due to the deep wavepacket-overlapings $\int dv \times \psi_\downarrow^\dagger(r)\psi_\uparrow(r)$.

For azimuthal and radial isoangles the respective *isounits* given above are,

$$\hat{I}_{\hat{\phi}} = n_1 n_2, \quad \hat{I}_{\hat{\theta}} = n_3 \tag{229}$$

An explicit example of the *isounits*, $\hat{I}_{s\,gr}$ and $\hat{I}_{t\,gr}$, in isominkowskian representation characterized by Schwartzchild's exterior metric for space and time for an observer in an exterior gravitational field are given by [11],

$$\hat{I}_{s\,gr} = (1 - 2M/r)^{1/2}\{1,\ 1,\ 1\},\ \ \hat{I}_{t\,gr} = (1 - 2M/r)^{-1/2},\ \ r > 2M \tag{230}$$

The subject matter that we have described in this section is by no means comprehensive but surely it is adequate for a beginner to get initiated into it.

3 Santilli's Isodual Mathematics

Santilli's *isodual mathematics* has been developed basically to describe the antimatter systems and was first introduced in 1985 [22, 23] and then in details in 1993 [16]. It is characterized by the map of the trivial unit $+1$ into its negative value,

$$I = +1 \longrightarrow I^d = -1 \tag{231}$$

and is termed as an *isodual map* and denoted by the superscript ^{d}. Then Santilli went ahead to reconstruct the entire conventional mathematics in such a way to admit -1, rather than $+1$, as a correct left and right multiplicative units.

As an example, we have the *isodual field*, $F^d(n^d, +^d, \times^d)$ [16]. $n^d = n \times I^d = -n$, where n is an ordinary number, equipped with the *isodual sum, isodual product, isodual norm*, etc.,

$$\begin{aligned} n^d +^d m^d &= (n+m) \times I^d = -n - m \\ n^d \times^d m^d &= n^d \times (-1) \times m^d = -n \times m \\ |\, n \,|^d &= |\, n \,| \times I^d = -\,|\, n \,| \end{aligned} \tag{232}$$

where $+$, $\times$ and $|\ ....\ |$ are the conventional operations of $F(n, +, \times)$.

Notice that as seen above the isodual fields admit a *negative definite norm*. It is then evident that *the isodual map reverses the sign of all quantities*, not only the charge, but also mass, time, energy, etc., with *the understanding that these negative quantities are now referred to negative units*. For instance, the *isodual time* is indeed given by negative value of time $t^d = t \times I_t^d = -t$, but referred to the *negative unit* $I_t^d = -1$ s. As a result, *motion backward in time referred to a negative unit of time is as causal as motion forward in time referred to a positive unit of time*. Similar results hold for negative masses referred to negative unit of mass, etc.

The *isodual mathematics* is given by *isodual numbers and fields, isodual spaces and geometries, isodual algebras and groups, isodual functional analysis*, and, inevitably, *isodual quantization* [2, 14, 15, 22, 23, 28–31]. With this new mathematical apparatus it was easy to prove [22, 23, 29, 31] that the isodual theory of antimatter yields a correct classical description which, after isodual quantization, yields genuine antiparticles.

With the above background it is advisable to describe *isodual mathematics* based on the *isomathematics platform*. This is described below in subsections of this section.

3.1 Motivation for the Isodual Mathematics

The motivations for the classical isodual theory of antimatter are rather numerous. First, there is the need to achieve a full equivalence in the treatment of matter and antimatter beginning at the classical level. In fact, far away galaxies and quasars may well be made up of antimatter. The

absence of a classical theory of antimatter therefore implies the evident impossibility of quantitative studies of this important astrophysical issue.

Second, the current gravitational treatment of antimatter is afflicted by a number of problematic aspects. Current theories are based on only one map from classical to operator settings, the naive quantization. Therefore, conventional classical representations of antimatter via positive energies do not yield antiparticles under quantization, but conventional particles with the mere reversal of the sign of the charge.

Third, there is a fundamental incompatibility between current theories of gravitation and unified gauge theories of electro-weak interactions which is due precisely to antimatter. In fact, current gravitational theories characterize antimatter via a positive-definite energy-momentum tensor, while electro-weak theories characterize antiparticles via negative energy states.

For additional motivation refer to the references [11, 29, 32]. The need for a systematic study aiming at a resolution of these issues is then beyond scientific doubts.

The classical theory of antimatter developed using *isodualmathematics* permits an apparent resolution of the above problematic aspects. In particular, the theory results are compatible with all known classical experimental data on antiparticles, those under electromagnetic interactions, since no conclusive experiment under gravitational interactions is available so far.

We finally indicate that the isodual theory of antimatter is deeply connected to a variety of pre-existing research. First, the isodual particles emerge as possessing a negative time precisely along the historical conception by Stueckelberg for antiparticle. Moreover, the equivalence of treatment between particles and antiparticles at all levels of study can be first seen in the Stueckelberg-Feynman path integral theory.

The isodual theory emerged from the identification of negative units in the antiparticle component of the conventional Dirac equation and the reconstruction of the theory with respect to that unit. *Isoduality* therefore provides a mere reinterpretation of Dirac's original notion of antiparticle, while leaving all numerical predictions under electro-weak interactions essentially unchanged.

The isodual discrete symmetries also turn out to be deeply linked to preexisting studies. As an example, the parity of antiparticles originally introduced by Bargmann, Wightman and Wigner (see [31]), when expressed

in the recent formulation by Ahluwalia, Johnson and Goldman (see [31]), turns out to be equivalent to isodual space inversions. Despite these similarities on physical grounds (which are evidently expected since all theories study the same physical problem), the reader should be aware that the isodual theory of antimatter of Santilli presented herein is mathematically inequivalent to pre-existing studies, as established by the fact that the latter are formulated on conventional spaces and fields, while the former is formulated on new spaces and fields.

3.2 Rudiments of Isodual Mathematics

Santilli's isodual mathematics [21] is the image of isomathematics under the anti-isomorphic *isodual map* of an arbitrary quantity,

$$\begin{aligned} A\left(t,\, r,\, p,\, \psi,\, \psi^{\dagger},\, ...\right) \;\rightarrow\; & A^{d}\left(t^{d},\, r^{d},\, p^{d},\, \psi^{d},\, \psi^{\dagger d},\, ...\right) \\ & = -A^{\dagger}\left(-t,\, -r^{\dagger},\, -p^{\dagger},\, -\psi^{\dagger},\, -\psi,\, ...\right) \end{aligned} \tag{233}$$

The basic quantity of isodual mathematics is then the *isodual isounit*, namely:

$$\hat{I}^{d} = -\hat{I}^{\dagger}\left(-t,\, -r^{\dagger},\, -p^{\dagger},\, -\psi^{\dagger},\, -\partial\psi^{\dagger},\, ...\right) = \frac{1}{\widehat{T}^{d}} < 0 \tag{234}$$

The *isodual isonumbers* are given by,

$$\hat{a}^{d} = -\hat{a}^{\dagger} = -\hat{I}^{\dagger} \times \hat{a}^{\dagger} \tag{235}$$

that follows the *associative and commutative isodual sum*

$$\hat{a}^{d} \hat{+}^{d} \hat{b}^{d} = \hat{a}^{d} + \hat{b}^{d} = -(\hat{a} + \hat{b})^{\dagger}, \tag{236}$$

the *distributive and associative isodual products*

$$\hat{a}^{d} \times^{d} \hat{b}^{d} = \hat{a}^{d} \times (\hat{I}^{d})^{-1} \times \hat{b}^{d} = -\hat{a}^{d} \times \hat{T} \times \hat{b}^{d} = -\hat{a}^{d} \hat{\times} \hat{b}^{d} = \hat{c}^{d}, \tag{237}$$

additive isodual isouniit $\hat{0}^{d} = \hat{0}$

$$\hat{a}^{d} +^{d} \hat{0}^{d} = \hat{0}^{d} +^{d} \hat{a}^{d} = \hat{a}^{d} \tag{238}$$

and *multiplicative isodual isounit* $\hat{I}^{d} = -\hat{I}^{\dagger}$,

$$\hat{a}^{d} \times^{d} \hat{I}^{d} = \hat{I}^{d} \times^{d} \hat{a}^{d} = \hat{a}^{d} \tag{239}$$

It is also evident that, for the consistency, *all operations on isonumbers must be subjected to isoduality when dealing with isodual isonumbers*. This implies *isodual powers*,

$$\left(\hat{a}^d\right)^{\hat{n}^d} = \hat{a}^d \hat{\times}^d \hat{a}^d \hat{\times}^d \hat{a}^d \hat{\times}^d \cdots\cdots \tag{240}$$

the *isodual isosquare root* is given as,

$$(\hat{a}^d)^{\widehat{(1/2)}^d} = \hat{\sqrt[2]{-\hat{a}^\dagger}}^\dagger, \quad (\hat{a}^d)^{\widehat{(1/2)}^d} \hat{\times}^d (\hat{a}^d)^{\widehat{(1/2)}^d} = \hat{a}^d, \quad (\hat{I}^d)^{\widehat{(1/2)}^d} = -\hat{i} \tag{241}$$

and *isodual quotient* reads as,

$$\hat{a}^d \hat{/}^d \hat{b}^d = -(a^\dagger / b^\dagger) = c^d \implies \hat{b}^d \hat{\times}^d \hat{c}^d = \hat{a}^d \tag{242}$$

The *isodual isonorm* is given as,

$$\hat{|}\hat{a}\hat{|}^{\hat{d}} = -\hat{|}\hat{a}\hat{|} = -\mid a \mid \times \hat{I} \tag{243}$$

that is always *negative definite*.

Whenever, one is required to use a *complex isonumber* then its *isodual* is generated by keeping in view that the *imaginary isounit* $\hat{i}$ is *isoselfdual*, viz. $\hat{i}^d = -\hat{i}^\dagger = -\overline{\hat{i}} = -(-\hat{i}) = \hat{i}$, where $\overline{\hat{i}}$ is the complex conjugate of $\hat{i}$, the isodual of an isonumber is obtained as $\hat{n}^d = -\hat{n}$ and $\hat{\times}^d = (\times\hat{T}\times)^d = \times\hat{T}^d\times = \times(-\hat{T})\times = -\hat{\times}$.

Now let $\hat{c} = (\hat{a} + \hat{i}\hat{\times}\hat{b})$ be a complex isofunction whose complex conjugate reads as $\overline{\hat{c}} = (\hat{a} - \hat{i}\hat{\times}\hat{b})$. Then the steps to compute isodual of $\hat{c}$ are

$$\hat{c}^d = (\hat{a} + \hat{i}\hat{\times}\hat{b})^d = \hat{a}^d + \hat{i}^d \hat{\times}^d \hat{b}^d = -\hat{a} + \hat{i}(-\times)(-\hat{b}) = -(\hat{a} - \hat{i}\hat{\times}\hat{b}) = -\overline{\hat{c}}$$

3.3 Isodual Functional Analysis

The *isodual functional analysis* has been studied by Kadeisvili [28] and Santilli [33] and others. Below we describe some of the functions.

1. *Isodual isotrigonometric functions* are defined as,

$$\widehat{\sin}^d \hat{\theta}^d = -\widehat{\sin}(-\hat{\theta}), \quad \widehat{\cos}^d \hat{\theta}^d = -\widehat{\cos}(-\hat{\theta}) \tag{244}$$

with a related basic property,

$$(\widehat{\sin}^d)^{\hat{2}^d} \hat{\theta}^d +^d (\widehat{\cos}^d)^{\hat{2}^d} \hat{\theta}^d = \hat{I}^d = -\hat{I} \tag{245}$$

2. the *isodual isohyperbolic functions* follows as,

$$\widehat{\sinh}^{\,d}\hat{\theta}^d = -\widehat{\sinh}(-\hat{\theta}), \quad \widehat{\cosh}^{\,d}\hat{\theta}^d = -\widehat{\cosh}(-\hat{\theta}) \tag{246}$$

with a related basic property,

$$(\widehat{\sinh}^{\,d})^{\hat{2}^d}\hat{\theta}^d - {}^d(\widehat{\cosh}^{\,d})^{\hat{2}^d}\hat{\theta}^d = \hat{I}^d = -\hat{I} \tag{247}$$

3. the *isodual isologarithm* reads as,

$$\hat{\log}^{\,d} n^d = -\log(-n) \tag{248}$$

4. and the *isodual isoexponentiation* gets defined as,

$$\left(\hat{e}^d\right)^{\hat{X}^d} = \hat{I}^d \times (\hat{e})^{\hat{T}^d \times \hat{X}^d} = -(\hat{e})^{\hat{X}} = -e^X \times \hat{I} \tag{249}$$

3.4 Isodual Differential and Integral Calculus

Contrary to a rather popular belief, the differential calculus is indeed dependent on the assumed unit. This property is not so transparent in the conventional formulation because the basic unit is the trivial number +1. However, the dependence of the unit emerges rather forcefully under its generalization.

The *isodual differential calculus*, first introduced by Santilli in [1], is characterized by the *isodual differential*

$$\hat{d}^d\hat{x}^d = \hat{I}^d \times \hat{d}\hat{x}^d = -\hat{d}\hat{x}^d \tag{250}$$

with corresponding *isodual derivative*,

$$\hat{\partial}^d\hat{/}^d\hat{\partial}^d\hat{x}^d = \hat{\partial}\hat{/}\hat{\partial}\hat{x}^d \tag{251}$$

and related *isodual properties*.

Note that the *conventional differentials are isoselfdual*, i.e.,

$$(dx)^d = d^d x^d \equiv dx, \tag{252}$$

but the *derivatives are not isoselfdual*,

$$(\partial f/\partial x)^d = -\left(\partial^d f^d /^d \partial^d x^d\right) \neq \partial f/\partial x \tag{253}$$

These properties explain why the isodual differential calculus remained undiscovered for centuries.

The *isodual integral calculus* easily follows from the above description,

$$\int^{\hat{d}} \hat{d}^{\hat{d}} \hat{f}^{\hat{d}} = \int^{\hat{d}} \hat{A}^{\hat{d}} \hat{\times}^{\hat{d}} \hat{d}^{\hat{d}} \hat{x}^{\hat{d}} = -\int \hat{A} \hat{\times} \hat{d}^{\hat{d}} \hat{x}^{\hat{d}} = -\int \hat{A} \hat{\times} \hat{d}\hat{x} = -\hat{I} \times \int A dx \quad (254)$$

provided the *isounit* is constant.

3.5 Four Directions of Time

It is believed that time has only two directions, the celebrated *Eddington's time's arrows* [34]. In reality, *time has four different directions* depending on whether motion is backward or forward and occurs in past or in future as illustrated in Figure 9. In turn, the correct use of all four different directions of time is mandatory, for instance, in serious studies of bifurcations.

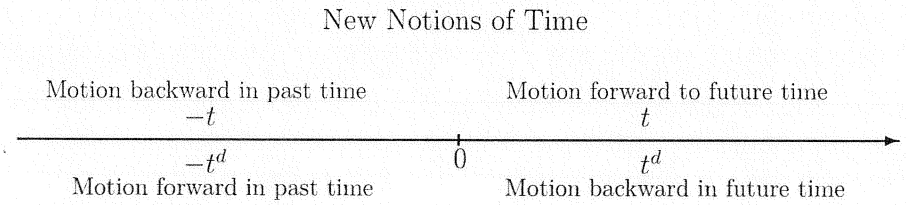


Figure 9: A schematic view of the four possible "time's directions".

It is evident that the theoretical physics of the 20^{th} century could not explain all four directions of time, since it possessed only one conjugation, time reversal, and this explains the reason the two remaining directions of time remained submerged.

It is equally evident that isoduality does indeed permit the representation of the two missing directions of time, thus further illustrating its need.

From above illustration the difference between the time reversal and isoduality gets clarified. Time reversal changes the direction of time while keeping the underlying space and units unchanged, while isoduality changes the direction of time while mapping the underlying space and units into different forms.

As depicted in Figure 9 t indicates *motion forward to future time*, $-t$ indicates *motion backward in past time*, t^d indicates *motion backward in future time* and $-t^d$ indicates *motion forward in past time*.

The reader is advised to develop other aspects of isodual mathematics corresponding to the ones described in section 2 and for more details of it the original literature should be referred.

4 Santilli's Genomathematics

The fundamental problem of isomathematics, despite achieving the broadening of unitary-canonical theories into nonunitary-noncanonical extensions, is that of the *origin of the irreversibility* of our macroscopic reality does not get admitted in terms of quantitative treatment because the Lie-Santilli isotheory is also structurally reversible (that is, the theory coincides with its time reversal image for reversible Hamiltonians and isounits).

The resolution of this insufficiency required suitable broadening of the Lie- Santilli isotheory. In turn, the achievement of an invariant formulation of the latter theory required the construction of a new mathematics that Professor Santilli formulated way back in 1978 [35] under the name of *genomathematics* (where the prefix geno now stands for axiom-inducing).

4.1 Genonumbers and Genoalgebra

The main idea of genomathematics is the selection of *two different generalized units* called *genounits*, the first $\hat{I}^{>}$ for the *ordered multiplication to the right* $A > B$, called *forward genoproduct*, and the second ${}^{<}\hat{I}$ for the *ordered multiplication to the left* $A < B$, called *backward genoproduct*, according to the general rules [15,16,35,36]:

$$\hat{I}^{>} = 1/\hat{S},\ A > B = A \times \hat{S} \times B,\ \hat{I}^{>} > A = A > \hat{I}^{>} = A, \tag{255}$$

$${}^{<}\hat{I} = 1/\hat{R},\ A < B = A \times \hat{R} \times B,\ {}^{<}\hat{I} < A = A <^{<} \hat{I} = A, \tag{256}$$

where $\hat{S}$ and $\hat{R}$ are respective *genotopic elements*, that is for the forward and backward multiplications respectively. Moreover, we have assumed,

$$A = A^{\dagger},\ B = B^{\dagger},\ \hat{R} = \hat{S}^{\dagger},\ \text{ and }\ \hat{R} \neq \hat{S} \tag{257}$$

hence we have by definition:

$$\hat{I}^{>} \neq {}^{<}\hat{I}$$

The *genounits* are conveniently generated from the isounit, $\hat{I}$, as follows,

$$\hat{I}^{>} = V \times \hat{I} \times V^{\dagger} > 0, \quad {}^{<}\hat{I} = W \times \hat{I} \times W^{\dagger} > 0 \tag{258}$$

Hence, we have,

$$\begin{aligned} \hat{S} &= (V \times \hat{I} \times V^{\dagger})^{-1} = V^{-1} \times \hat{I}^{-1} \times (V^{\dagger})^{-1} \\ &= V^{-1} \times \hat{T} \times (V^{\dagger})^{-1} = V^{-1} \hat{\times} (V^{\dagger})^{-1} > 0 \end{aligned} \tag{259}$$

Similarly,

$$\begin{aligned} \hat{R} &= (W \times \hat{I} \times W^{\dagger})^{-1} = W^{-1} \times \hat{I}^{-1} \times (W^{\dagger})^{-1} \\ &= W^{-1} \times \hat{T} \times (W^{\dagger})^{-1} = W^{-1} \hat{\times} (W^{\dagger})^{-1} > 0 \end{aligned} \tag{260}$$

That is we have opted,

$$V \times V^{\dagger} \neq 1, \quad W \times W^{\dagger} \neq 1$$

The *genonumbers* get generated as,

$$\hat{A}^{>} = A \times \hat{I}^{>}, \quad {}^{<}\hat{A} = A \times {}^{<}\hat{I} \tag{261}$$

The *genosum* get expressed as,

$$\hat{A}^{>} \hat{+}^{>} \hat{B}^{>} = \hat{A}^{>} + \hat{B}^{>}, \quad {}^{<}\hat{A} \, {}^{<}\hat{+} \, {}^{<}\hat{B} = {}^{<}\hat{A} + {}^{<}\hat{B} \tag{262}$$

consequently the *additive genounits* $\hat{0}^{>}$ and ${}^{<}\hat{0}$ coincide with the ordinary 0 and the *forward* and *backward genoproducts* are such that $\hat{I}^{>}$ and ${}^{<}\hat{I}$ are the correct left and right *multiplicative genounits*,

$$\hat{I}^{>} > \hat{A}^{>} = \hat{A}^{>} > \hat{I}^{>} = \hat{A}^{>}, \quad {}^{<}\hat{I} < {}^{<}\hat{A} = {}^{<}\hat{A} < {}^{<}\hat{I} = {}^{<}\hat{A} \tag{263}$$

For each element $\hat{A}^{>}$ and ${}^{<}\hat{A}$ there exist respectively $(\hat{A}^{>})^{-\hat{I}^{>}}$ and $({}^{<}\hat{A})^{-{}^{<}\hat{I}}$ and are called *forward genoinverse* and *backward genoinverse* respectively such that,

$$\hat{A}^{>} > (\hat{A}^{>})^{-\hat{I}^{>}} = \hat{I}^{>}, \quad {}^{<}\hat{A} < ({}^{<}\hat{A})^{-{}^{<}\hat{I}} = {}^{<}\hat{I} \tag{264}$$

The *genosums* are *commutative*,

$$\hat{A}^{>}\,\hat{+}^{>}\,\hat{B}^{>} \;=\; \hat{B}^{>}\,\hat{+}^{>}\,\hat{A}^{>}, \quad {}^{<}\hat{A}\,{}^{<}\hat{+}\,{}^{<}\hat{B} \;=\; {}^{<}\hat{B}\,{}^{<}\hat{+}\,{}^{<}\hat{A} \tag{265}$$

and *associative*,

$$\begin{aligned}\left(\hat{A}^{>}\,\hat{+}^{>}\,\hat{B}^{>}\right)\,\hat{+}^{>}\,\hat{C}^{>} \;&=\; \hat{A}^{>}\,\hat{+}^{>}\,\left(\hat{B}^{>}\,\hat{+}^{>}\,\hat{C}^{>}\right)\\ \left({}^{<}\hat{A}\,{}^{<}\hat{+}\,{}^{<}\hat{B}\right)\,{}^{<}\hat{+}\,{}^{<}\hat{C} \;&=\; {}^{<}\hat{A}\,{}^{<}\hat{+}\,\left({}^{<}\hat{B}\,{}^{<}\hat{+}\,{}^{<}\hat{C}\right)\end{aligned} \tag{266}$$

The *genotopic lifting* of *iso-multiplication of iso-numbers* get expressed as,

$$\begin{aligned}\hat{A}\hat{\times}\hat{B} \;\longrightarrow\; & V\times\left(\hat{A}\hat{\times}\hat{B}\right)\times V^{\dagger}\\ &= V\times\hat{A}\times\times V^{\dagger}\times(V^{\dagger})^{-1}\hat{\times}(V)^{-1}\times V\times\hat{B}\times\times V^{\dagger}\\ &= \hat{A}^{>}\times V^{-1}\times\hat{T}\times(V^{\dagger})^{-1}\times\hat{B}^{>}\\ &= \hat{A}^{>}\times\hat{S}\times\hat{B}^{>}=\hat{A}^{>}\hat{\times}^{>}\hat{B}^{>}\end{aligned} \tag{267}$$

and

$$\begin{aligned}\hat{A}\hat{\times}\hat{B} \;\longrightarrow\; & W\times\left(\hat{A}\hat{\times}\hat{B}\right)\times W^{\dagger}\\ &= W\times\hat{A}\times\times W^{\dagger}\times(W^{\dagger})^{-1}\hat{\times}(W)^{-1}\times W\times\hat{B}\times\times W^{\dagger}\\ &= {}^{<}\hat{A}\times W^{-1}\times\hat{T}\times(W^{\dagger})^{-1}\times{}^{<}\hat{B}\\ &= {}^{<}\hat{A}\times\hat{R}\times{}^{<}\hat{B}={}^{<}\hat{A}\,{}^{<}\hat{\times}\,{}^{<}\hat{B}\end{aligned} \tag{268}$$

Thus we see that,

$$\hat{\times}^{>}=\times\hat{S}\times=\times V^{-1}\times\hat{T}\times(V^{\dagger})^{-1}\times \tag{269}$$

$${}^{<}\hat{\times}=\times\hat{R}\times=\times W^{-1}\times\hat{T}\times(W^{\dagger})^{-1}\times \tag{270}$$

The *genotopic lifting* of an *iso-quotient of iso-numbers* reads as,

$$\begin{aligned}\hat{A}\hat{\div}\hat{B} \;\longrightarrow\; & V\times(\hat{A}\hat{\div}\hat{B})\times V^{\dagger}=V\times\left(\frac{\hat{A}}{\hat{B}}\times\hat{I}\right)\times V^{\dagger}\\ &= \frac{V\times\hat{A}\times V^{\dagger}}{V\times\hat{B}\times V^{\dagger}}\times V\times\hat{I}\times V^{\dagger}\\ &= \frac{\hat{A}^{>}}{\hat{B}^{>}}\times\hat{I}^{>}=\hat{A}^{>}\,\hat{\div}^{>}\,\hat{B}^{>}\end{aligned} \tag{271}$$

or alternatively,

$$\begin{aligned}\hat{A}\hat{\div}\hat{B} \quad \longrightarrow \quad & V\times(\hat{A}\hat{\div}\hat{B})\times V^{\dagger}=V\times(\hat{A}(\div\times\hat{I})\hat{B})\times V^{\dagger}\\ & =\Big((V\times\hat{A}\times V^{\dagger})\div(V\times\hat{B}\times V^{\dagger})\Big)\times V\times\hat{I}\times V^{\dagger}\\ & =\hat{A}^{>}\,\hat{\div}^{>}\,\hat{B}^{>}\end{aligned}\tag{272}$$

That means we have for *genotopic division*,

$$\hat{\div}^{>}=\div\times\hat{I}^{>},\quad \hat{/}^{>}=(/)\times\hat{I}^{>}\tag{273}$$

and

$$^{<}\hat{\div}=\div\times{}^{<}\hat{I},\quad {}^{<}\hat{/}=(/)\times{}^{<}\hat{I}\tag{274}$$

Forward and Backward genoproducts are associative,

$$\begin{aligned}\hat{A}^{>}\,>\,\Big(\hat{B}^{>}\,>\,\hat{C}^{>}\Big)\;&=\;\Big(\hat{A}^{>}\,>\,\hat{B}^{>}\Big)\,>\,\hat{C}^{>}\\ {}^{<}\hat{A}\,<\,\Big({}^{<}\hat{B}\,<\,{}^{<}\hat{C}\Big)\;&=\;\Big({}^{<}\hat{A}\,<\,{}^{<}\hat{B}\Big)\,<\,{}^{<}\hat{C}\end{aligned}\tag{275}$$

but not necessarily commutative,

$$\Big(\hat{A}^{>}\,>\,\hat{B}^{>}\Big)\neq\Big(\hat{B}^{>}\,>\,\hat{A}^{>}\Big),\quad \Big({}^{<}\hat{A}\,<\,{}^{<}\hat{B}\Big)\neq\Big({}^{<}\hat{B}\,<\,{}^{<}\hat{A}\Big)\tag{276}$$

4.2 Genofunctions

The lifting of isofunctions $\hat{f}(\hat{r})$ on $\hat{F}$ into the *forward and backward genofunctions* $\hat{f}^{>}(\hat{r}^{>})$ and ${}^{<}\hat{f}({}^{<}\hat{r})$ on $\hat{F}^{>}$ and ${}^{<}\hat{F}$, respectively are achieved as follows.

$$\hat{f}^{>}\,(\hat{x}^{>})=f(x)\times\hat{I}^{>},\quad {}^{<}\hat{f}\,({}^{<}\hat{x})=f(x)\times{}^{<}\hat{I}\tag{277}$$

For example, the *genoexponentiation* reads as,

$$\begin{aligned}(\hat{e}^{>})^{\hat{X}^{>}}\;&=\;\Big(e^{\hat{X}^{>}\times\hat{S}}\Big)\times\hat{I}^{>}=\hat{I}^{>}\times\Big(e^{\hat{X}^{>}\times\hat{S}}\Big)\\ ({}^{<}\hat{e})^{{}^{<}\hat{X}}\;&=\;\Big(e^{\hat{R}\times{}^{<}\hat{X}}\Big)\times{}^{<}\hat{I}\;=\;{}^{<}\hat{I}\times\Big(e^{\hat{R}\times{}^{<}\hat{X}}\Big)\end{aligned}\tag{278}$$

and with consequential genotopies of transforms and functional analysis at large see ref. [15] and references cited therein.

4.3 Genodifferentials and Genointegration

The *forward* and *backward genodifferentials and derivatives* are given as,

$$\begin{aligned} \hat{d}^{>}\hat{x}^{>} &= \hat{S}_x^{>} \times d(x \times \hat{I}_x^{>}) \xrightarrow{\hat{I}^{>}=const} dx \\ {}^{<}\hat{d}^{<}\hat{x} &= {}^{<}\hat{R}_x \times d(x \times^{<} \hat{I}_x) \xrightarrow{{}^{<}\hat{I}=const} dx \end{aligned} \tag{279}$$

and

$$\hat{\partial}^{>}(\hat{/}^{>})\hat{\partial}^{>}\hat{r}_i^{>} = \hat{S}_{ij}^{>} \times \partial/\partial\hat{r}_j^{>}, \quad {}^{<}\hat{\partial}({}^{<}\hat{/}){}^{<}\hat{\partial}\,{}^{<}\hat{r}_i = {}^{<}\hat{R}_{ij} \times \partial/\partial\,{}^{<}\hat{r}_j. \tag{280}$$

Then the *forward and backward genotopic kronecker delta* is given by,

$$\hat{\partial}^{>}\hat{r}_i^{>}(\hat{/}^{>})\hat{\partial}^{>}\hat{r}_j^{>} = \hat{\delta}_{ij}^{>} = \delta_{ij} \times \hat{I}^{>}, \quad {}^{<}\hat{\partial}\,{}^{<}\hat{r}_i({}^{<}\hat{/}){}^{<}\hat{\partial}\,{}^{<}\hat{r}_j = {}^{<}\hat{\delta}_{ij} = \delta_{ij} \times^{<} \hat{I} \tag{281}$$

The broader genomathematics is then given by:

1. The lifting of isotopologies, isogeometries, etc. into the dual forward and backward genotopic forms; and
2. The lifting of the Lie-Santilli isotheory into the genotheory, including the genotopies of the various aspects, such as universal enveloping associative algebras for ordered product to the right and to the left, etc. (see ref. [15] and references cited therein).

4.4 Lie-Admissible (Genotopic) Operator Mechanics

Let $|\ \psi^{>}\rangle$, $|\ \phi^{>}\rangle$, $|^{<}\ \psi\rangle$, $|^{<}\ \phi\rangle$, ..., be *genostates* and the forward and backward genounits be $\hat{I}^{>} = Q^{-1}$ and ${}^{<}\hat{I} = P^{-1}$. For the sake of simplicity we have omitted the depiction of the basic isonature of the quantities that is by not using the cap on the quantities.

The corresponding *genoinner products* and *genonormalization* read as,

$$\begin{aligned} \langle\phi^{>}\ |>|\ \psi^{>}\rangle &= \langle\phi^{>}\ |\times Q\times|\ \psi^{>}\rangle \times \hat{I}^{>} \\ \langle^{<}\phi\ |<|^{<}\ \psi\rangle &= \langle^{<}\phi\ |\times P\times|^{<}\ \psi\rangle \times^{<} \hat{I} \end{aligned} \tag{282}$$

and

$$\langle\psi^{>}\ |>|\ \psi^{>}\rangle = \hat{I}^{>}, \quad \langle^{<}\psi\ |<|^{<}\ \psi\rangle = {}^{<}\hat{I}, \quad \hat{I}^{>} = \left({}^{<}\hat{I}\right)^{\dagger} \tag{283}$$

where

$$\langle|>|\rangle = \langle| \times Q \times |\rangle \times \hat{I}^{>}, \quad \langle|<|\rangle = \langle| \times P \times |\rangle \times^{<} \hat{I} \tag{284}$$

This explains well that the original axiomatic structure on genotopic lifting remains unchanged, that is, $\langle| \times |\rangle \to \langle| \times P \times |\rangle$ but the underlying unit is lifted by *inverse amount*, $1 \to P^{-1}$.

Hence, we have,

$$\begin{aligned} \langle\phi \mid \times \mid \psi\rangle &\equiv \langle\phi \mid \times \mid \psi\rangle \times P \times P^{-1} = \langle\phi \mid \times P \times \mid \psi\rangle \times \times P^{-1} \\ &= \langle\phi \mid <\mid \psi\rangle \times^{<} \hat{I} \\ \langle\phi \mid \times \mid \psi\rangle &\equiv \langle\phi \mid \times \mid \psi\rangle \times Q \times Q^{-1} = \langle\phi \mid \times Q \times \mid \psi\rangle \times \times Q^{-1} \\ &= \langle\phi \mid >\mid \psi\rangle \times \hat{I}^{>} \end{aligned} \tag{285}$$

Thus it also follows that,

$$\hat{I}^{>} >\mid \psi^{>}\rangle = \mid \psi^{>}\rangle, \quad {}^{<}\hat{I} <\mid^{<} \psi\rangle = \mid^{<} \psi\rangle, \tag{286}$$

The *genoexpectation values* are given by,

$$< A^{>} > = \frac{\langle {}^{<}\psi \mid > A^{>} >\mid \psi^{>}\rangle}{\langle {}^{<}\psi \mid >\mid \psi^{>}\rangle} \tag{287}$$

A most important feature of the Lie-Santilli genotheory is its *form invariance*. Now once again revert back to original representations.

$$V \times V^{\dagger} \neq 1, \; V =^{<} \hat{V} \times \hat{R}^{1/2}, \; V \times V^{\dagger} =^{<} \hat{V} <^{<} \hat{V}^{\dagger} =^{<} \hat{V}^{\dagger} <^{<} \hat{V} =^{<} \hat{I}, \tag{288}$$

$$\begin{aligned} W \times W^{\dagger} &\neq 1, \; W = \hat{W}^{>} \times \hat{S}^{1/2}, \\ W \times W^{\dagger} &= \hat{W}^{>} > \hat{W}^{>\dagger} = \hat{W}^{>\dagger} > \hat{W}^{>} = \hat{I}^{>}, \end{aligned} \tag{289}$$

under which we have indeed the following forward genoinvariance laws [37],

$$\hat{I}^{>} \to \hat{I}'^{>} = \hat{W}^{>} > \hat{I}^{>} > \hat{W}^{>\dagger} = \hat{I}^{>}, \tag{290}$$

$$\hat{A} > \hat{B} \to \hat{W}^{>} > (\hat{A} > \hat{B}) > \hat{W}^{>\dagger} = \hat{A}' > \hat{B}', \tag{291}$$

$$\begin{aligned} \hat{H}^{>} > |\rangle &= \hat{E}^{>} > |\rangle = E \times |\rangle \to \hat{W}^{>} > \hat{H}^{>} > |\rangle \\ &= \hat{H}'^{>} > |\rangle' = \hat{W}^{>} > \hat{E}^{>} > |\rangle = E \times |\rangle', \end{aligned} \tag{292}$$

with corresponding rules for the backward and classical counterparts.

The above rules confirm the achievement of the *invariance of the numerical values of genounits, genoproducts and genoeigenvalues*, thus permitting physically consistent applications.

By recalling again that Lie's theory is at the foundation of all the contemporary sciences, the Lie-Santilli genotheory has permitted an additional structural generalization of classical and quantum isomechanics, isosuperconductivity and isochemistry into their genotopic coverings.

Intriguingly, the product $\hat{A} < \hat{B} - \hat{B} > \hat{A} = \hat{A} \times \hat{R} \times \hat{B} - \hat{B} \times \hat{S} \times \hat{A}$, $\hat{R} \neq \hat{S}$, is manifestly non-Lie on conventional spaces over conventional fields, yet it becomes fully antisymmetry and Lie when formulates on the bimodule of the respective envelopes to the left and to the right, $\{^{<}\hat{A}, \hat{A}^{>}\}$ (explicitly, the numerical values of $\hat{A} < \hat{B} = \hat{A} \times \hat{R} \times \hat{B}$ computed with respect to $^{<}\hat{I} = 1/\hat{R}$ is the same as that of $\hat{A} > \hat{B} = \hat{A} \times \hat{S} \times \hat{B}$ when computed with respect to $\hat{I}^{>} = 1/\hat{S}$) (see ref. [15] and references cited therein).

More importantly, genotheories have permitted a resolution of the historical problem of the origin of irreversibility via its reduction to the ultimate possible layers of nature, such as particles in the core of a star. The interested reader can find the invariant genotopic formulations of: Newtons equations, Hamiltons equations with external terms, quantization for open-irreversible systems, operator theory of open-irreversible systems in references cited in [15].

4.5 Genoisodual Mathematics

As one develops the isodual mathematics of isomathematics to tackle antimatter systems on the same lines to tackle time irreversible processes of antimatter systems the isodual branch of genomathematics is needed. This aspect we leave as an exercise for the readers because isodual of isomathematics has already been described in section 3.

4.6 Genotime's Four Arrows

From the above described Lie-Santilli isotheory and its isodual it is evident that there are *four* directions of time. Recall that Eddington's arrow of time identifies only two directions of time, namely future and past [34].

The identification of two hither to unnoticed directions of time became possible on adopting *isoduality* by Santilli. Notice that the traditional *time reversal* changes only the direction of time while keeping the underlying space and units *unchanged* whereas the *isoduality* changes the direction of time while mapping the underlying space and units into different forms. Therefore, in reality, *time has four different directions* depending on whether the motion is forward or backward and occurs in the future or in the past. Herein the forward and backward motions are referred to that of matter and antimatter respectively whereas while spelling out the time indicating the particular motion the matter and antimatter is not specified to assert that it is a more general classification of time's arrows.

Recall from section 3.5 where we have described four directions of time [21], namely:

1. Time $\hat{t}$ is taken to represent the *motion forward to future time.*
2. Time $-\hat{t}$ is taken to represent the *motion backward in past time.*
3. Time $\hat{t}^d$ is taken to represent the *motion forward in past time.*
4. Time $-\hat{t}^d$ is taken to represent the *motion backward in future time.*

As noted by Santilli this classification based on iso- and isoduality lacks the irreversibility involved in natural processes and hence he formulated the genotheory. Thus the irreversible descriptions are inherent in genomechanics and in its isodual. The four explicit directions of time shown in Figure 10 now get the following representations, namely [1,4]:

1. Time $\hat{t}^{>}$ is taken to represent the *motion forward in future time.*
2. Time $^{<}\hat{t}$ is taken to represent the *motion backward in past time.*
3. Time $^{<}\hat{t}^d$ is taken to represent the *motion backward in future time.*
4. Time $\hat{t}^{d>}$ is taken to represent the *motion forward in past time.*

GENOTIMES

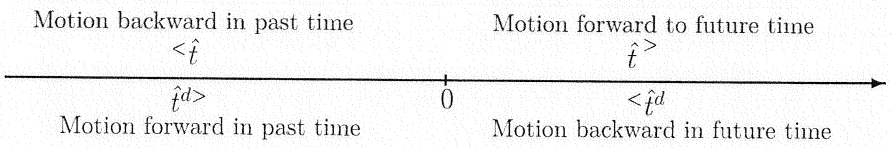


Figure 10: A schematic view of the four possible "Genotime's directions". It coincides that of Figurer 9 when forward and backward genounits coincide.

Note that *the four possible "time arrows" are represented by the time genounits,* $\hat{I}_{\hat{t}}^{>}$, $<\hat{I}_{\hat{t}}$, $\hat{I}_{\hat{t}}^{d>}$, $<\hat{I}_{\hat{t}}^{d}$, *while time remains the ordinary real quantity*. Hence, the above four times are generated from the real time, t, as follows:

$$\hat{t}^{>} = t\hat{I}_{\hat{t}}^{>}, \quad \hat{I}_{\hat{t}}^{>} = (\hat{T}_{\hat{t}}^{>})^{-1} \neq \left(\hat{I}_{\hat{t}}^{>}\right)^{\dagger} \quad \text{for motion forward to future times, } >$$

$$<\hat{t} = <\hat{I}_{\hat{t}} t, \qquad <\hat{I}_{\hat{t}} = \left(\hat{I}_{\hat{t}}^{>}\right)^{\dagger} \quad \text{for motion backward in past times, } <$$

$$<\hat{t}^{d} = <\hat{I}_{\hat{t}}^{d} t, \qquad <\hat{I}_{\hat{t}}^{d} = \left(\hat{I}_{\hat{t}}^{d>}\right)^{\dagger} \quad \text{for motion backward in future times, } <$$

$$\hat{t}^{d>} = t\hat{I}_{\hat{t}}^{d>}, \quad \hat{I}_{\hat{t}}^{d>} = (\hat{T}_{\hat{t}}^{d>})^{-1} \neq \left(\hat{I}_{\hat{t}}^{d>}\right)^{\dagger} \quad \text{for motion forward in past times, } >$$

The non-Hermiticity of Santilli genounits and its isodual is prescribed by

$$\hat{I}_{\hat{t}}^{>} \neq <\hat{I}_{\hat{t}}, \quad \hat{I}_{\hat{t}}^{d>} \neq <\hat{I}_{\hat{t}}^{d}$$

and the time in isodual genospace increases in reverse direction than that in the genospace is ensured by

$$\hat{t}^{d>} = -\hat{t}^{>}, \ <\hat{t}^{d} = -<\hat{t} \quad \text{implying} \quad \hat{I}_{\hat{t}}^{d>} = -\hat{I}_{\hat{t}}^{>}, \ <\hat{I}_{\hat{t}}^{d} = -<\hat{I}_{\hat{t}}$$

At the same time the following intriguing and far reaching aspect emerges in biology. In the above description we have strictly used isodual theories for the sole representation of antimatter. However, Illert [20] has shown that the representation of the bifurcations in sea shells requires the use of all four directions of time. Hence in this context isodual theory is lifted to take care of additional directions needed in biology but thereby no features of antimatter would get involved. The latter aspect is an additional illustration of the complexity of biological system. In fact, the occurrence of the said complexity implies that the intrinsic time of a seashell, that is, the time perceived by a sea shells as a living organism, is so complex and so far remained beyond our comprehension. Alternatively, we can say that the complexity of hypertimes is intended to reflect the complexity of biological systems [9].

The present author has developed a comprehensive geno-thermodynamic theory of irreversible processes of systems consisting of matter [6, 38] he aims to extend the formalism to the biological system. But the constitutive theory of biological systems using new mathematics has not been accomplished to the desired level. However, our attempts at thermodynamic level perhaps may show a path to accomplish it.

5 Lie-Santilli Hypertheory

By no means *genotheories* are sufficient to represent in entirety intricacies of nature, e.g., because they are unable to represent *biological structures* such as a cell or a sea shell. The latter systems are indeed open-nonconservative-irreversible, yet they possess a structure dramatically more complex than that of a nonconservative Newtonian system. A study of the issue has revealed that the limitation of genotheories is due to their single-valued character.

As an illustration, mathematical treatments complemented with computer visualization [20] have established that the shape of seashells can be well described via the conventional single-valued three-dimensional Euclidean space and geometry according to the empirical perception of our three Eustachian tubes. However, the same space and geometry are basically insufficient to represent the growth in time of seashells. In fact, computer visualization shows that, under the exact imposition of the Euclidean axioms, seashells first grow in time in a distorted way and then

crack.

Illert [20] showed that a minimally consistent representation of the sea shells growth in time requires six dimensions. But sea shells exist in our environment and can be observed via our three-dimensional perception. The solution of this apparent dichotomy was proposed [20] via multi-valued hypermathematics essentially characterized by the relaxation of the single-valued nature of the genounits while preserving their nonsymmetric character (as a necessary condition to represent irreversible events), according to the rules [15,36]:

$$\hat{I}^{>} = \left\{ \hat{I}_1^{>}, \hat{I}_2^{>}, \hat{I}_3^{>}, ... \right\} = 1/\hat{S}, \tag{293}$$

$$\begin{aligned} A > B \;\; = \;\; & \left\{ A \times \hat{S}_1 \times B, A \times \hat{S}_2 \times B, A \times \hat{S}_3 \times B, ... \right\}, \\ & \hat{I}^{>} > A = A > \hat{I}^{>} = A \times I, \end{aligned} \tag{294}$$

$$^{<}\hat{I} = \left\{ ^{<}\hat{I}_1, ^{<}\hat{I}_2, ^{<}\hat{I}_3, ... \right\} = 1/\hat{R}, \tag{295}$$

$$\begin{aligned} A < B \;\; = \;\; & \left\{ A \times \hat{R}_1 \times B, A \times \hat{R}_2 \times B, A \times \hat{R}_3 \times B, ... \right\}, \\ & ^{<}\hat{I} < A = A <^{<} \hat{I} = I \times A, \end{aligned} \tag{296}$$

$$A = A^{\dagger}, \; B = B^{\dagger}, \; \hat{R} = \hat{S}^{\dagger} \tag{297}$$

All aspects of the bimodular genotheories admit a unique, and significant extension to the above hyperstructures and their explicit form is omitted here for brevity [15,36]. The expression of the theory via *weak equalities and operations* was first studied by Santilli and Vougiouklis in Ref. [14].

The *isodual hypermathematics* has been developed on the same lines as that in the case of genomathematics. The reader is advised to refer the original literature for example see [13]. Hence corresponding four sets of time's arrows emerge that has been depicted in Figure 11.

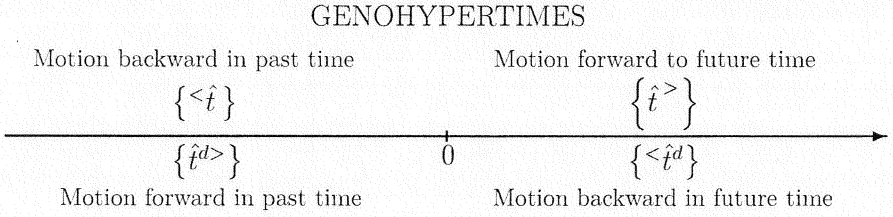


Figure 11: A schematic view of the four sets of possible "Genohypertime's directions". It coincides that of Figurer 9 when all forward and backward genhyperounits coincide.

6 Time's Arrows in New Thermodynamics

The first attempt to reframe thermodynamics using Santilli's new mathematics [1–5] is that due to Dunning-Davies [8–10, 24, 39]. He developed iso- and geno-thermodynamics for matter and their isoduals for antimatter systems. For the sake of self-sufficiency let us first describe his new thermodynamics and then discuss the time's arrows associated with them.

6.1 Iso-Thermodynamics for Matter

The first application of geno-, iso- and isodual- mathematics to thermodynamics has been described in Refs. [8–10, 24, 39], and can be summarized as follows.

$$TdS = T \times dS \to \widehat{T} \,\widehat{\times}\, \widehat{dS} \to T \times \widehat{I} \times \widehat{K} \times \widehat{dS} \to T \times \widehat{dS}, \tag{298}$$

where

$$I \to \widehat{I} = \frac{1}{\widehat{K}} > 0$$

is the Hermitean *isounit* and $\widehat{K}$ the *isotopic element* at the foundation of Santilli's isomathematics. Then

$$T \times dS \to T \times \widehat{dS} = T \times \widehat{I} \times d\left(S \times \widehat{I}\right) = \widehat{I} \times TdS. \tag{299}$$

Hence, it follows immediately that,

$$\begin{aligned} dQ &= dU + pdV \to \widehat{dQ} = \widehat{dU} + \widehat{p} \widehat{\times} \widehat{dV} \to \widehat{I} \times dQ \\ &= \widehat{I} \times (dU + p\, dV) \Rightarrow dQ = dU + pdV \end{aligned}$$

and

$$dQ = TdS \to \widehat{dQ} = \widehat{T} \,\widehat{\times}\, \widehat{dS} \to \widehat{I} \times dQ = \widehat{I} \times TdS \Rightarrow dQ = TdS.$$

This then demonstrates that, within the iso-mathematical framework, the equations representing the first and second laws of thermodynamics hold in their familiar forms. This also confirms that all standard thermodynamic relations and equations would retain their familiar forms in iso-mathematics' framework, namely:

$$TS = U + pV - \mu N,$$

the Gibbs-Duhem equation:

$$SdT - Vdp + Nd\mu = 0,$$

the Gibbs relation:

$$dS = T^{-1}dU + pT^{-1}dV - \mu T^{-1}dN,$$

and so on.

6.2 Geno-Thermodynamics for Matter

The conformity of the standard thermodynamic relations and equations with geno-mathematics is demonstrated as follows.

$$TdS = dU + pdV \to T^{>} > d^{>}S^{>} = d^{>}U^{>} + p^{>} > d^{>}V^{>} \tag{300}$$

where $I^{>} = 1/T'^{>}$ is the generally non-Hermitean *genounit* and $T'^{>}$ is the *genotopic element* at the foundation of Santilli's genomathematics. The above expression can be written

$$\begin{aligned} (TI^{>})\, I^{>-1} \left[I^{>-1} d\,(SI^{>}) \right] &= TdS \\ &= I^{>-1} d\,(UI^{>}) \\ &\quad + (pI^{>})\, I^{>-1} \left[I^{>-1} d\,(VI^{>}) \right] \\ &= dU + p\, dV. \end{aligned}$$

If the geno-unit depends on the local variables, we have,

$$dS \to d^{>}S^{>} = I^{>\,-1} d\,(SI^{>}) = dS + SI^{>\,-1} dI^{>}$$

and similarly for dQ and dW. Hence we have the following transformations,

$$T^{>} > d^{>}S^{>} = d^{>}U^{>} + p^{>} > d^{>}V^{>} \to$$
$$\to TdS + TSI^{>\,-1} dI^{>} =$$
$$= dU + UI^{>\,-1} dI^{>} + pdV + pVI^{>\,-1} dI^{>}$$

Thus if $I^{>}$ depends on the local thermodynamic variables the standard thermodynamic relation doesn't get regenerated (because we do not have $TS = U + pV$) instead we have,

$$TdS - G(I^{>\,-1} dI^{>}) = dU + pdV$$

where we have used the the standard thermodynamic relation $G = U + pV - TS$. Now further, as proposed by Dunning-Davies [9] if we opt the genounit as a function of entropy S, say $I^{>} = e^{f(S)}$ then we have,

$$T\left(1 - \frac{G}{T}\frac{\partial f(S)}{\partial S}\right) dS = dU + pdV$$

Therefore, if we identify, T^{*} as the projection of the genotemperature, $T^{>}$, on the traditional thermodynamic space given by,

$$T^{*} = T\left(1 - \frac{G}{T}\frac{\partial f(S)}{\partial S}\right)$$

then the thermodynamic equation would read as,

$$T^{*} dS = dU + pdV$$

This necessitates the recognition of a new definition of temperature in genothermodynamics. It seemingly opens a new vista in genothermodynamics that should be investigated in details including other possible options.

Let us now consider again the possibility to express genounit in terms of a function of entropy, S. As stated above on using $I^{>} = \exp(f(S))$ the Gibbs relation gets transformed to,

$$T\left(1 - \frac{G}{T}\frac{\partial f(S)}{\partial S}\right) dS = dU + pdV \qquad (301)$$

Thus we see that by adopting $I^{>} = \exp(f(S))$ one obtains $\frac{G}{T}\frac{\partial f(S)}{\partial S}$ as the irreversibility contribution in the Gibbs relation. However, when we use $I^{>} = \exp(f(S))$ in the corresponding genotopically lifted Gibbs-Duhem equation, namely:

$$S^{>} > d^{>}T^{>} - V^{>} > d^{>}p^{>} + n^{>} > d^{>}\mu^{>} = 0$$

the final result is,

$$(SdT - Vdp + nd\mu) + \left(U\frac{\partial f(S)}{\partial S}\right) dS = 0 \tag{302}$$

which is the Gibbs-Duhem equation counterpart of eq.(301) but it is not acceptable because it includes a term containing dS. A Gibbs-Duhem equation cannot have an extensive quantity as a variable. The reason for the production of this unacceptable result lies in the fact that we have directly genotopically lifted the final thermodynamic equations. This is similar to the celebrated proposal of local equilibrium assumptio (LEA) where the traditional Gibbs relation is directly lifted to nonequilibrium domain. Thus the message is that we should not venture to genotopically lift final thermodynamic relations. Instead we should go for an ab initio approach.

Let us proceed on these lines. The present author has shown elsewhere [6] (Geno-Nonequilibrium Thermodynamics. II.) that for not far away from equilibrium situations we have the following genotopic relation of the per unit genomass geno Gibbs function, $\mathcal{G}^{>}$,

$$\mathcal{G}^{>} = \mu^{>} + \alpha^{>}_{\mathbf{q}^{>}} > \mathbf{q}^{>} \cdot \mathbf{q}^{>} + \alpha^{>}_{\mathbf{\Pi}^{>}} > \mathbf{\Pi}^{>} : \mathbf{\Pi}^{>} \tag{303}$$

where we have used the following expressions of $\alpha^{>}_{\mathbf{q}^{>}}$ and $\alpha^{>}_{\mathbf{\Pi}^{>}}$, namely:

$$\alpha^{>}_{\mathbf{q}^{>}} = \frac{2^{>}}{5^{>} > n^{>} > p^{>} > k^{>}_{B} > T^{>}}, \qquad \alpha^{>}_{\mathbf{\Pi}^{>}} = \frac{1^{>}}{2^{>} > p^{>} > m^{>} > n^{>}} \tag{304}$$

Now on using the standard relation of the geno-Gibbs function, $\mathcal{G}^{>} = u^{>} + p^{>} > v^{>} - T^{>} > s^{>}$, we obtain the following genononequilibrium equation,

$$u^{>} = T^{>} > s^{>} - p^{>} > v^{>} + \mu^{>} + \alpha^{>}_{\mathbf{q}^{>}} > \mathbf{q}^{>} \cdot \mathbf{q}^{>} + \alpha^{>}_{\mathbf{\Pi}^{>}} > \mathbf{\Pi}^{>} : \mathbf{\Pi}^{>} \tag{305}$$

However, on genomathematical manipulations the preceding equation gets transformed as,

$$u \times I^{>} = (sT - pv + \mu + \alpha_{\mathbf{q}}\mathbf{q} \cdot \mathbf{q} + \alpha_{\mathbf{\Pi}}\mathbf{\Pi} : \mathbf{\Pi}) \times I^{>} \tag{306}$$

The above result is perfectly in tune with the fundamental requirement of the genotopy that is the left and right hand side of eq.(306) are the multiplication of the conventional numbers by the genounit, $I^{>}$.

However, at the same time it also reveals that the *genothermodynamic* expressions/relations do not depend on the explicit expression of the genounit used. This is not a surprising outcome because in the geno-GPITT that has been described by the present author elsewhere [6] there arisen no need to consider the interactions at fm or less distances.

In view of the above revelation the basic proposal of Dunning-Davies of expressing the genounit in terms of some thermodynamic function would be appropriate only in those systems requiring the consideration of interactions at fm or less. To do this one has to appropriately express the geno-Gibbs function taking care of the said interactions including the associated thermodynamic irreversibility via hadronic mechanics. This is an interesting as well as a basic aspect needs to be investigated further.

6.3 Isodual-Thermodynamics for Antimatter

Dunning-Davies [8, 39] also demonstrated that Santilli's anti-Hermitean Isodualities, denoted with the upper symbol $(^d)$ lead us to the thermodynamics of antimatter for which the standard thermodynamic relations retain their forms. The demonstration follows the following steps.

$$\begin{aligned} dQ \to d^dQ^d &= dQ,\ dU \to d^dU^d = dU, \\ dW \to d^dW^d &= dW,\ dS \to d^dS^d = dS \end{aligned} \tag{307}$$

as we have the following isoduality transformations,

$$\begin{aligned} &df \to d^df^d = -d\left[-f\right],\ +1 \to -1 = 1^d, \\ &\therefore\ 1^d \times^d n = (-1) \times (-1)n = n \end{aligned}$$

that produces:

$$\begin{aligned} &Q \to Q^d = -Q,\ U \to U^d = -U,\ W \to W^d = -W, \\ &S \to S^d = -S,\ T \to T^d = -T. \end{aligned}$$

On using above transformations the following gets established:

$$dQ \to d^dQ^d = dQ,\ dU - dW \to d^dU^d - d^dW^d = dU - dW.$$

That is, the First law of thermodynamics is *isoselfdual*. Similarly,

$$dQ \to d^dQ^d = dQ,\ TdS \to T^d \times^d d^dS^d = (-T) \times (-1)dS = TdS.$$

Therefore, $TdS = dU - dW$ and $TdS = dU + pdV - \mu dN$ are *isoselfdual*. That is they are equally valid for antimatter too perhaps including the Gibbs-Duhem equation, $SdT - Vdp + Nd\mu = 0$.

6.4 Time's Arrows Associated with New Thermodynamics

The above developed new thermodynamics by Dunning-Davies [8–10, 24, 39] obviously on the face of it is a description having four time directions as described in Figures 9 and 10 respectively for iso- and its isodual thermodynamics taken together and geno- and its isodual thermodynamics taken together. That is they are associated with the iso- and geno- times depicted in Figures 9 and 10.

7 Time's Arrows. Genothermodynamics Version

Now with the development of geno-GPITT, an exclusive genononequilibrium thermodynamics, described elsewhere [6,7,38] by the present author the concept of *genothermodynamic directions of time* originate. The origin of the *genothermodynamic directions of time* follows as under. For this purpose we first briefly describe the laws of thermodynamic based genothermodynamic framework for systems in nonequilibrium and then describe the systems having irreversibility only due to the occurrence of chemical reactions at finite rates described elsewhere by the present author [6]. However, before that the relevant rudiments of manipulations of Clausius' inequality to arrive at the *genoentropy* function has been described below.

7.1 Genotopic Lifting of Clausius' Inequality and Generation of Genoentropy Function

It is well known that for every thermodynamic nonequilibrium situation the traditional definition $dQ = TdS$ is not valid [40–44]. Hence, we need

to first define corresponding entropy function using the mathematical expressions that takes care of the existing irreversibility. The best candidate for this is Clausius' inequality. The aspects that we initially propose to handle, which is expected to streamline the subject of nonequilibrium thermodynamics as recently reiterated by the present author [41], are listed below.

1. Re-express using geno-mathematics Clausius' inequality [45], namely:

$$\oint_{irr} \frac{dQ}{T_R} < 0 \qquad \text{(closed systems)} \tag{308}$$

where T_R is the temperature of thermally interacting heat reservoir with which the system exchanges dQ amount of heat, and its generalized version [41–43, 46], namely:

$$\oint_{irr} \left[\frac{1}{T_R(t)} \frac{dQ}{dt}\right] dt$$

$$\equiv \oint_{irr} \left[\int_A -\frac{\mathbf{Q}(\mathbf{A},t)}{T(\mathbf{A},t)} \cdot \mathbf{dA}\right] dt < 0 \quad \text{(closed systems)} \tag{309}$$

where $\mathbf{Q}(\mathbf{A},t)$ is the heat flux density at the surface area coordinate $\mathbf{A}$ and at time t, $\mathbf{dA}$ is the differential area vector on the surface of the system, A is the total surface area of the closed system and $T(\mathbf{A},t)$ is temperature at $\mathbf{A}$ and t [47].

2. Generate the *genoentropy* function appropriate for irreversible processes from the so modified versions of Clausius' inequality both for spatially uniform and non-uniform systems.

3. Incorporate the local level fluid-dynamical and global level internal energy balance equations that have been lifted to genotopic level and generate genomathematics based corresponding Gibbs relations.

4. In doing so the dictates of the *"universe of operations of genothermodynamics"* as described elsewhere [6] which is based on the revealations about the *"universe of operations of thermodynamics"* by Bridgman [48, 49] and the present author [40, 41, 50] would get inherently followed.

7.1.1 Geno-Clausius' Inequality

In geno-mathematics eqs.(308) and (299) would get lifted as [46],

$$\oint^{>} \frac{d^{>}Q^{>}}{T_R} < 0 \qquad \text{(closed systems)} \tag{310}$$

$$\oint^{>} \left[\frac{1^{>}}{T_R} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>}$$

$$\equiv \oint^{>} \left[\int_{A^{>}}^{>} -\frac{\mathbf{Q}^{>}(\mathbf{A}^{>})}{T^{>}(\mathbf{A}^{>})} \cdot \mathbf{d}^{>}\mathbf{A}^{>}\right] > d^{>}t^{>} < 0 \quad \text{(closed systems).} \tag{311}$$

Thus eqs.(310) and (311) are the *geno-Clausius inequalities*. Notice that the subscript irr below the cyclic integration sign has been omitted because it has become redundant as well as the explicit time dependence of heat flux vector and temperature function is not required to be shown. Moreover, as T_R is the temperature of the heat reservoir which by definition remains at equilibrium and hence it cannot be lifted to genotopic level. However, by definition the temperature in the second part of eq.(311) is that at the surface of the system in nonequilibrium [47] therefore it has to be lifted to genotopic level that on the said lifting has been denoted as $T^{>}(\mathbf{A}^{>})$. This temperature obviously follows from the *generalized zeroth law of thermodynamics* [6, 41, 47]. Also notice that for the sake of simplicity we have dropped the hat (ˆ) over the symbols as has been followed by Dunning-Davies in his geno-thermodynamics [8, 39].

7.1.2 Genoentropy for Spatially Non-uniform Systems

Now from eq.(310) on following mathematical manipulations adopted by Clausius [45], Eu [51, 52] and Eu and García-Colín [53] we define the *geno-uncompensation function*, $N^{>}$, as,

$$N^{>} = -\oint^{>} \frac{d^{>}Q^{>}}{T_R} = -\oint^{>} \frac{1^{>}}{T_R} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} > d^{>}t^{>} > 0 \tag{312}$$

Thus we obtain the following dictate of the second law of thermodynamics in terms of *geno-uncompensation function*, $N^{>}$, namely:

$$N^{>} \geq 0. \tag{313}$$

That is $N^{>} = 0$ is the equation of a reversible cycle. Now we treat $N^{>}$ as an independent quantity [53] to express it as,

$$N^{>} = \oint^{>} \frac{d^{>}N^{>}}{d^{>}t^{>}} > d^{>}t^{>} \geq 0 \tag{314}$$

Again as the *geno-uncompensation function* appears only on account of irreversibility, $d^{>}N^{>}$ cannot be negative and should vanish on each reversible segment of the cycle [42, 51], therefore, from eq.(314) we have

$$d^{>}N^{>} > 0 \qquad \text{as} \qquad N^{>} > 0. \tag{315}$$

Notice that the positive definiteness of $d^{>}N^{>}$ further adds to the mathematical description of unidirectionality of time's arrow during the irreversible evolution of a system. In other words, eq.(315) imparts *time's arrow a genothermodynamic status*.

On combining eqs.(312) and (314) we obtain

$$\oint^{>} \left[\frac{1^{>}}{T_{R}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} + \frac{d^{>}N^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>} = 0. \tag{316}$$

As cyclic integral of eq.(316) vanishes we obtain

$$\frac{d^{>}S^{>}}{d^{>}t^{>}} = \frac{1^{>}}{T_{R}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} + \frac{d^{>}N^{>}}{d^{>}t^{>}}. \tag{317}$$

Thus we see that in eq.(317) $S^{>}$ is the entropy of a nonequilibrium state that henceforth we will term as *genoentropy*. In the limit of $d^{>}N^{>} \to 0$, we have $S^{>}$(genoentropy) $\to S$(the entropy of an equilibrium state). Now as $d^{>}N^{>}/d^{>}t^{>} > 0$, from eq.(317) we obtain:

$$\frac{d^{>}S^{>}}{d^{>}t^{>}} > \left(\frac{1^{>}}{T_{R}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right) \tag{318}$$

which is nothing else but the explicit expression of *differential geno-Clausius inequality in time rate form*, which is traditionally expressed as [54],

$$dS > \frac{dQ}{T}. \tag{319}$$

It is worth highlighting that Clausius simply stated the inequality of eq.(319) but he never derived it. Hence, he was silent about the nature of the entropy function appearing on the left hand side of it that was pointed out

by Meixner as late as in the year 1973 [55]. The integrated form of Clausius' inequality was derived by Clausius himself, namely $\Delta S > \int_B^A \frac{dQ}{T}$ that definitely contains entropy S of an equilibrium state.

However, eq.(317) contains T_R hence the first term on its right hand side cannot be transformed as such to a local level description for a spatially non-uniform systems. For that purpose we need to use the corresponding expression from eq.(311), namely:

$$\frac{1^>}{T_R} > \frac{d^>Q^>}{d^>t^>} \equiv \int_{A^>}^{>} -\frac{\mathbf{Q}^>(\mathbf{A}^>)}{T^>(\mathbf{A}^>)} \cdot \mathbf{d}^>\mathbf{A}^> \tag{320}$$

Now on applying Gauss theorem we obtain,

$$\frac{1^>}{T_R} > \frac{d^>Q^>}{d^>t^>} \equiv \oint_{V^>}^{>} \left[-\nabla^> \cdot \left(\frac{\mathbf{q}^>(\mathbf{r}^>)}{T^>(\mathbf{r}^>)} \right) + \mathbf{q}^>(\mathbf{r}^>) \cdot \nabla^> \left(\frac{1^>}{T^>(\mathbf{r}^>)} \right) \right] > d^>V^> \tag{321}$$

that reduces to,

$$\frac{1^>}{T_R} > \frac{d^>Q^>}{d^>t^>} \equiv \int_{V^>}^{>} -\frac{1^>}{T^>(\mathbf{r}^>)} > (\nabla^> \cdot \mathbf{q}^>(\mathbf{r}^>)) > d^>V^>. \tag{322}$$

The other two terms of eq.(317) get easily transformed to

$$\frac{d^>S^>}{d^>t^>} = \int_{V^>}^{>} \rho^> > \frac{d^>s^>}{d^>t^>} > d^>V^> \tag{323}$$

and

$$\frac{d^>N^>}{d^>t^>} = \int_{V^>}^{>} \rho^> > \frac{d^>\mathcal{N}^>}{d^>t^>} > d^>V^> \geq 0 \tag{324}$$

wherein we have used the standard fluid dynamical inter-conversions after genotopic lifting, namely:

$$S^> = \int_{V^>}^{>} \rho^> > s^> > d^>V^>, \qquad N^> = \int_{V^>}^{>} \rho^> > \mathcal{N}^> > d^>V^>, \qquad \text{etc.}$$

where $\rho^>$ is the geno-mass density and $s^>$, $\mathcal{N}^>$, etc. are the per unit geno-mass geno-quantities at a given position and time $t^>$.

On substituting eqs. (322)-(324) into eq.(317) we obtain at the local level,

$$\rho^{>} > \frac{d^{>}s^{>}}{d^{>}t^{>}} = -\frac{1^{>}}{T^{>}} > (\nabla^{>} \cdot \mathbf{q}^{>}) + \rho^{>} > \frac{d^{>}\mathcal{N}^{>}}{d^{>}t^{>}}. \tag{325}$$

In eq.(325) we have omitted the position dependencies of the quantities for the sake of simplicity but it remains implied. The same is followed now onward in all mathematical representations. Notice that, in contrast to $d^{>}N^{>}/d^{>}t^{>} > 0$ there we have at the local level $d^{>}\mathcal{N}^{>}/d^{>}t^{>} \lesseqgtr 0$ because each interior tiny local pocket is an open system. Hence in general the local rate of geno-uncompensation function gets expressed as:

$$\rho^{>} > \frac{d^{>}\mathcal{N}^{>}}{d^{>}t^{>}} = -\nabla^{>} \cdot \mathbf{J}^{>}_{\mathcal{N}^{>}} + \sigma^{>}_{\mathcal{N}^{>}} \tag{326}$$

where $\mathbf{J}^{>}_{\mathcal{N}^{>}}$ is the *flux density of the geno-uncompensation function* and $\sigma^{>}_{\mathcal{N}^{>}}$ is the *source strength of geno-uncompensation function*. Moreover, when we integrate eq.(326) for the total volume of a closed system one obviously have,

$$\int_{V^{>}}^{>} (\nabla^{>} \cdot \mathbf{J}^{>}_{\mathcal{N}^{>}}) > d^{>}V^{>} = 0 \tag{327}$$

and hence, we obtain from eqs.(325), (326) and (327) we have,

$$\frac{d^{>}N^{>}}{d^{>}t^{>}} = \int_{V^{>}}^{>} \sigma^{>}_{\mathcal{N}^{>}} > d^{>}V^{>} \geq 0 \tag{328}$$

which then establishes $\sigma^{>}_{\mathcal{N}^{>}} \geq 0$ and hence eq.(326) gets expressed as,

$$\rho^{>} > \frac{d^{>}\mathcal{N}^{>}}{d^{>}t^{>}} + \nabla^{>} \cdot \mathbf{J}^{>}_{\mathcal{N}^{>}} = \sigma^{>}_{\mathcal{N}^{>}} \geq 0 \tag{329}$$

Now on combining eqs.(325) and (329) we have at local level,

$$\rho^{>} > \frac{d^{>}s^{>}}{d^{>}t^{>}} + \nabla^{>} \cdot \left(\frac{\mathbf{q}^{>}}{T^{>}} + \mathbf{J}^{>}_{\mathcal{N}^{>}}\right) = \mathbf{q}^{>} \cdot \nabla^{>} \left(\frac{1^{>}}{T^{>}}\right) + \sigma^{>}_{\mathcal{N}^{>}} \geq 0. \tag{330}$$

Notice that in eq.(330) we have derived the *geno-Clausius-Duhem inequality*, namely:

$$\rho^{>} > \frac{d^{>}s^{>}}{d^{>}t^{>}} + \nabla^{>} \cdot \mathbf{J}^{>}_{s^{>}} = \sigma^{>}_{s^{>}} \geq 0 \tag{331}$$

whence the *genoentropy flux density*, $\mathbf{J}^{>}_{s^{>}}$, and the *genoentropy source strength*, $\sigma^{>}_{s^{>}}$, are now given by,

$$\mathbf{J}^{>}_{s^{>}} = \left(\frac{\mathbf{q}^{>}}{T^{>}} + \mathbf{J}^{>}_{\mathcal{N}^{>}}\right) \tag{332}$$

$$\sigma^{>}_{s^{>}} = \mathbf{q}^{>} \cdot \nabla^{>}\left(\frac{1^{>}}{T^{>}}\right) + \sigma^{>}_{\mathcal{N}^{>}} \geq 0. \tag{333}$$

Thus it gets established that $\sigma^{>}_{s^{>}}$ is the *rate of genoentropy production per unit volume*.

At this juncture one may ask that – *what new has been achieved in the preceding description*? – because if one compares preceding equations with the corresponding expressions/equations of reference [41] he would find that they are identically same except for the use of superscript (>) to the quantities in the above description. To answer this virtual objection, one needs to recall that the equations of reference [41], and for that matter of all non-nonequilibrium thermodynamics, are based on the traditional differential calculus/Newtonian mechanics. In those equations/expressions, the time reversal is effected by substituting $-t$ for t. However, when this is effected the signs of $\frac{ds}{dt}$ changes, but in view of the elaboration elsewhere by the present author (Geno-Nonequilibrium Thermodynamics. I) [7] no unambiguous change or invariance of signs of entropy exchange and that of entropy production results. This then reveals the inadequacy of traditional differential calculus/Newtonian mechanics to unambiguously take care of irreversibility of the processes. On the other hand, in geno-mathematics the time reversal is effected by basing the mathematical description on ${}^{<}I$ and $<$ instead of $I^{>}$ and $>$ where by definition $I^{>} \neq {}^{<}I$. That is the forward and reverse trajectories are not the same, perfectly in tune with the reality.

7.1.3 Genoentropy for Spatially Uniform Closed Systems

In this case the *geno-Clausius inequality* reads as,

$$\oint^{>}\left[\frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>} < 0 \tag{334}$$

Now by following the steps described above in the case of spatially non-uniform system we generate the *genoentropy function* appropriate for spa-

tially uniform systems. The *geno-uncompensation function*, $N^{>}$, gets defined as,

$$N^{>} = -\oint^{>} \left[\frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>} \geq 0 \tag{335}$$

Herein too we obtain the following dictate of the second law of thermodynamics in terms of *geno-uncompensation function*, $N^{>}$, namely:

$$N^{>} \geq 0. \tag{336}$$

That is $N^{>} = 0$ is the equation of a reversible cycle. Now we treat $N^{>}$ as an independent quantity [53] to express it as,

$$N^{>} = \oint^{>} \frac{d^{>}N^{>}}{d^{>}t^{>}} > d^{>}t^{>} \geq 0 \tag{337}$$

Again as the *geno-uncompensation function* appears only on account of irreversibility, $d^{>}N^{>}$ cannot be negative and would vanish on each reversible segment of the cycle [42, 51], therefore, from eq.(337) we have

$$d^{>}N^{>} \geq 0 \qquad \text{as} \qquad N^{>} \geq 0. \tag{338}$$

Notice that the positive definiteness of $d^{>}N^{>}$ further adds to the mathematical description of unidirectionality of *geno- time's arrow* during the irreversible evolution of a system. In other words, eq.(338) imparts *geno-time's arrow a genothermodynamic status*.

On combining eqs.(335) and (337) we obtain

$$\oint^{>} \left[\frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} + \frac{d^{>}N^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>} = 0. \tag{339}$$

The vanishing of cyclic integral of eq.(339) provides,

$$\frac{d^{>}S^{>}}{d^{>}t^{>}} = \frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} + \frac{d^{>}N^{>}}{d^{>}t^{>}}. \tag{340}$$

Thus we see that in eq.(340) $S^{>}$ is the corresponding *genoentropy*. Of course, the *genoentropy* of eq.(340) can easily be and directly obtained on imposing the condition of spatial uniformity on eq.(317).

In the limit of $d^{>}N^{>} \to 0$, we have $S^{>}$(genoentropy) $\to S$(the entropy of an equilibrium state). Now as $d^{>}N^{>}/d^{>}t^{>} > 0$, from eq.(340) we obtain:

$$\frac{d^{>}S^{>}}{d^{>}t^{>}} > \left(\frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right) \tag{341}$$

which is nothing else but the explicit *differential geno-Clausius' inequality in time rate form*, which is traditionally expressed as [54],

$$dS > \frac{dQ}{T}.$$

7.2 Geno-DeDonderian Thermodynamic Framework

In the preceding subsection we have developed the necessary equations of *genoentropy*. Out of them eq.(340) is appropriate for a spatially uniform closed system. The thermodynamic irreversibility that can exist in such systems is that due to the occurrence of chemical conversions at finite rates. However, for the sake of simplicity we begin by considering the case of a single chemical reaction proceeding at a finite rate. Let us consider the following typified single chemical reaction,

$$\nu'_{\mathrm{A}}\mathrm{A} + \nu'_{\mathrm{B}}\mathrm{B} + \nu'_{\mathrm{C}}\mathrm{C} + \cdots \longrightarrow \nu_{\mathrm{M}}\mathrm{M} + \nu_{\mathrm{N}}\mathrm{N} + \nu_{\mathrm{O}}\mathrm{O} + \cdots \tag{342}$$

where A, B, etc. are the chemical symbols of chemical reactants, M, N, etc. are the chemical symbols of chemical products, ν'_{A}, etc. are the stoichiometric coefficients of reactants and ν_{M} etc. are the stoichiometric coefficients of products. Further, we recall that with the progress of chemical reaction the mole numbers of reactants and products change at a finite rate. This change gets described by Dalton's law of constant proportions that mathematically reads as [54],

$$dn_k = \nu_k d\xi \tag{343}$$

that in genotime rate form reads as,

$$\frac{d^{>}n_k^{>}}{d^{>}t^{>}} = \left(\nu_k^{>} > \frac{d^{>}\xi^{>}}{d^{>}t^{>}}\right) \tag{344}$$

where $\frac{d^{>}\xi^{>}}{d^{>}t^{>}}$ is the *geno-rate of reaction* and is taken positive in forward direction which uniquely determines the changes of mole numbers of all reactants and products as reaction proceeds.

Now as assumed there is only one irreversible process and that is a single chemical reaction proceeding at a finite rate within the system hence it is the sole source of thermodynamic irreversibility. The *geno-uncompensation*

function appears in our description only if there exists thermodynamic irreversibility and hence the former gets quantified in this case through the *geno-rate of reaction* or to be specific we logically have,

$$\frac{d^{>}N^{>}}{d^{>}t^{>}} \propto \frac{d^{>}\xi^{>}}{d^{>}t^{>}} \quad \Rightarrow \quad \frac{d^{>}N^{>}}{d^{>}t^{>}} = \frac{\mathcal{A}^{>}}{T^{>}} > \frac{d^{>}\xi^{>}}{d^{>}t^{>}} \geq 0 \tag{345}$$

where the proportionality constant is taken as $\frac{\mathcal{A}^{>}}{T^{>}}$ and $\mathcal{A}^{>}$ is termed as the *geno-chemical affinity* of the chemical reaction under consideration. Because by convention in the forward direction (in the direction the chemical reaction is proceeding) the *geno-rate of reaction* has been taken as positive then the positive definite sign of eq.(345) can be adhered only if we have $\mathcal{A}^{>} > 0$. That is, $\mathcal{A}^{>} > 0$ *determines the genothermodynamic forward direction of a chemical reaction*. It is important to keep in mind that because of the structure of genomathematics one *cannot* have negative sign of the *geno-rate of reaction*. Thus the above *genothermodynamical condition* gets stated as below,

$$\frac{\mathcal{A}^{>}}{T^{>}} > 0 \quad \text{implies} \quad \frac{d^{>}\xi^{>}}{d^{>}t^{>}} > 0 \tag{346}$$

That is $\mathcal{A}^{>} > 0$ determines the *genothermodynamic forward direction* of a chemical reaction.

Next the above geno- description gets extend to a spatially uniform system within which more than one chemical reactions are proceeding at finite rates. Thus now we have,

$$\frac{d^{>}N^{>}}{d^{>}t^{>}} = \sum_{\gamma} \frac{(\mathcal{A}^{>})^{\gamma}}{T^{>}} > \frac{d^{>}(\xi^{>})^{\gamma}}{d^{>}t^{>}} \geq 0 \tag{347}$$

where the superscript γ denotes the γ^{th} chemical reaction. In deriving eq.(347) we have used the following expression of Dalton's law of constant proportions, namely:

$$\frac{d^{>}n_{k}^{>}}{d^{>}t^{>}} = \sum_{\gamma} \left((\nu_{k}^{>})^{\gamma} > \frac{d^{>}(\xi^{>})^{\gamma}}{d^{>}t^{>}} \right) \tag{348}$$

to the extent of identifying a single parameter for each chemical reaction that uniquely quantifies corresponding mole number changes.

7.3 Production and Exchange of Genoentropy

From the above discussion it gets revealed that the rate of change of genoentropy is determined by two types of contributions, namely its exchange and production. This we express using the expression of genoentropy production in the Belgian school notation [54] as,

$$\frac{d^{>}S^{>}}{d^{>}t^{>}} = \frac{d_e^{>}S^{>}}{d^{>}t^{>}} + \frac{d_i^{>}S^{>}}{d^{>}t^{>}} \tag{349}$$

where the *rate of exchange of genoentropy* reads as,

$$\frac{d_e^{>}S^{>}}{d^{>}t^{>}} = \frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}} \gtrless 0 \tag{350}$$

and the *rate of production of genoentropy* reads as,

$$\frac{d_i^{>}S^{>}}{d^{>}t^{>}} = \frac{d^{>}N^{>}}{d^{>}t^{>}} \geq 0 \tag{351}$$

Thus the *genoentropy of a system can change on two and only two counts* one by the mechanism of exchange of it with its surroundings and the second by its production within the system.

In the present case, the existing irreversibility is only because the chemical reactions are proceeding at finite rates, the genothermodynamic expression of rate of genoentropy production is obtained using eqs.(347) and (351) as,

$$\frac{d_i^{>}S^{>}}{d^{>}t^{>}} = \sum_{\gamma} \frac{(\mathcal{A}^{>})^{\gamma}}{T^{>}} > \frac{d^{>}(\xi^{>})^{\gamma}}{d^{>}t^{>}} \geq 0 \tag{352}$$

7.4 Coupled Chemical Reactions

It is natural to ask if there are more than one chemical reactions proceeding at finite rates then is it imperative that for all chemical reactions their chemical affinities and rates have to be positive definite individually? The obvious answer is not necessarily so. When two processes get coupled then the contribution of one out of them may be negative to the rate of genoentropy production provided the magnitude of the net genoentropy production still remains positive. This we elaborate by considering that

within the system two chemical reactions are proceeding at finite rates. The rate of genoentropy production from eq.(352) will read as,

$$\frac{d_i^{>}S^{>}}{d^{>}t^{>}} = \left(\frac{(\mathcal{A}^{>})^1}{T^{>}} > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}} + \frac{(\mathcal{A}^{>})^2}{T^{>}} > \frac{d^{>}(\xi^{>})^2}{d^{>}t^{>}}\right) > 0 \qquad (353)$$

From eq.(353) we get the following inequality, namely:

$$\left(\frac{d^{>}(\xi^{>})^2}{d^{>}t^{>}}\right) > -\left(\frac{(\mathcal{A}^{>})^1}{(\mathcal{A}^{>})^2} > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}\right) \qquad (354)$$

Thus when $(\mathcal{A}^{>})^1$ and $\frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}$ are positive (that is this chemical reaction is proceeding in the direction dictated by genothermodynamics) then even if $(\mathcal{A}^{>})^2$ is negative still $\frac{d^{>}(\xi^{>})^2}{d^{>}t^{>}}$ can be positive provided the positive definite sign of rate of genoentropy production is maintained. That is the second chemical reaction in reality proceeds in the opposite direction then that dictated by the second law of thermodynamics. Hence the chemical reaction 1 is termed as *coupling chemical reaction* and the chemical reaction 2 is termed as *coupled chemical reaction*. This amazing revelation about the allowed freedom within the strict dictates of the second law of thermodynamics get best exemplified by biochemical processes. The two examples are given below.

Examples:

Let us consider two biochemical cases [54, 56], namely:

(1) The case of biological burning of sugar that couples the production of urea:

$$\frac{1}{6}\,C_6H_{12}O_6 + O_2 \longrightarrow CO_2 + H_2O \qquad \mathcal{A}_1^{>} = 115.0 \times I^{>}\ \text{kcal/mol}$$

$$2\,NH_3 + CO_2 \longrightarrow (NH_2)_2CO + H_2O \qquad \mathcal{A}_2^{>} = -11.0 \times I^{>}\ \text{kcal/mol}$$

$$\therefore \frac{d^{>}(\xi^{>})^2}{d^{>}t^{>}} > \left(\frac{115}{11} > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}\right) \approx 10 > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}.$$

(2) The case of biological hydrolysis of Creatine phosphate that couples the ATP (the energy store house of biological systems) synthesis:

$$\text{Creatine Phosphate} + H_2O \longrightarrow \text{Creatine} + P \qquad \mathcal{A}_1^{>} = 43.5 \times I^{>}\ \text{kJ/mol}$$

$$\mathrm{ADP} + \mathrm{P} \longrightarrow \mathrm{ATP} + \mathrm{H_2O} \qquad \mathcal{A}_2^{>} = -39.8 \times I^{>}\ \mathrm{kJ/mol}$$

$$\therefore \frac{d^{>}(\xi^{>})^2}{d^{>}t^{>}} > \left(\frac{43.5}{39.8} > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}\right) \approx 1.1 > \frac{d^{>}(\xi^{>})^1}{d^{>}t^{>}}.$$

In the above two representative examples the urea production and the ATP synthesis are individually thermodynamically unfavourable but the non-equilibrium thermodynamics and more appropriately the *genononequilibrium thermodynamics* explains *how and why in biological systems their synthesis is unbelievably efficient*.

In view of the above revelation one can split rate of genoentropy production in two types as follows,

$$\frac{d_i^{>}S^{>}}{d^{>}t^{>}} = \left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\alpha} + \left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\beta} > 0 \tag{355}$$

with

$$\left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\alpha} > 0 \tag{356}$$

clubbing in it all positive terms, that is of *all coupling chemical reactions*, and

$$\left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\beta} < 0 \tag{357}$$

accommodating all negative terms, of *all coupled chemical reactions*, in it.

Thus eq.(356) maintains the thermodynamic irreversibility and eq.(357) tries to maintain the order. The observations that the natural evolutions are the *ordered irreversible phenomena* that gets lucidly explained by the mechanism of genothermodynamic coupling of chemical reactions.

At this stage we also *need to convince ourselves* that whether the genomathematics branch, which is based on $t^{>}$ and hence all quantities $Z^{>}$, permits describing in a same frame (i) a chemical reaction proceeding in a direction dictated by the second law of thermodynamics and (ii) the one not following the said dictate because it is coupled with the former is a sound looking explanation. The answer to this question is an obviously no. This is so because the rational behind the development and use of genomathematics gets flouted. Thus we have now reached a stage where we are led to rationally structure the genothermodynamics by identifying genothermodynamic times and corresponding arrows of time as dictated

by the second law of thermodynamics. Moreover, it is hoped that this proposed structuring then could be easily extended to consider coupled- biochemical, biophysical and as such the biological evolutionary processes. This vital and crucial exercise we perform in the following subsubsection.

7.5 Genothermodynamic Times and Time's Arrows

The *universe of operations of thermodynamics* both with respect to the minimum volume and the minimum time duration of measurements has been described earlier [41, 44, 50]. Thus the *universe of operations of genothermodynamics* gets described as,

$$(\delta^{>}V^{>})_{lowest} \leq (\delta^{>}V^{>})_{(geno-therm)} \leq (\delta^{>}V^{>})_{highest}.$$

and

$\infty > \widehat{\tau}^{>} \geq \widehat{\tau}^{>}_{geno-therm}$	corresponds to the "Universe of operations of genononequilibrium thermodynamics"
$\widehat{\tau}^{>} < \widehat{\tau}^{>}_{geno-therm}$	corresponds to the natural fluctuational domain of genothermodynamic space.

From the above examples we learn that there are processes proceeding in accordance of the dictates of the second law of thermodynamics and those proceeding against the dictates of the second law of thermodynamics, and each one of them could be forward or backward moving ones. Thus in all there are four types of processes proceeding in *genothermodynamic space*. Now we identify correspondingly *four genothermodynamic time's arrows* as follows,

1. Time $\widehat{t}^{>}_{geno-therm}$ is taken to represent the *genothermodynamic motion forward in conformity with the dictates of the second law of thermodynamics*.

2. Time $^{<}\widehat{t}_{geno-therm}$ is taken to represent the *genothermodynamic motion backward in conformity with the dictates of the second law of thermodynamics*.

3. Time $^{<}\widehat{t}^{d}_{geno-therm}$ is taken to represent the *genothermodynamic motion backward against the dictates of the second law of thermodynamics*.

4. Time $\widehat{t}^{d>}_{geno-therm}$ is taken to represent the *genothermodynamic motion forward against the dictates of the second law of thermodynamics.*

with $\widehat{t}^{d>}_{geno-therm} = -\widehat{t}^{>}_{geno-therm}$, $<\widehat{t}^{d}_{geno-therm} = -<\widehat{t}_{geno-therm}$. These thermodynamic arrows of time have been depicted in Figure 12.

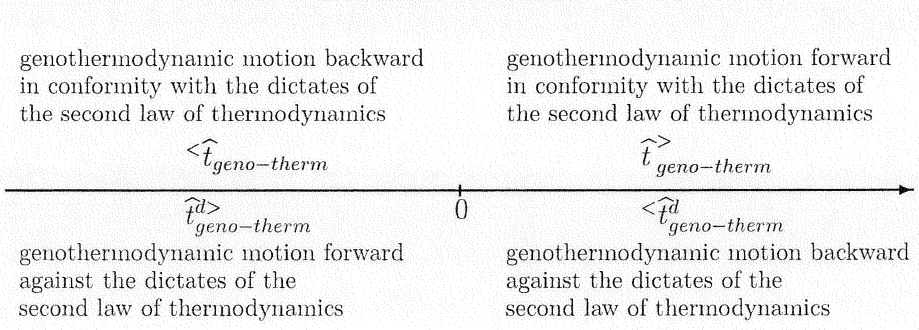

GENOTHERMODYNAMIC TIMES
genothermodynamic motion backward in conformity with the dictates of the second law of thermodynamics
genothermodynamic motion forward in conformity with the dictates of the second law of thermodynamics
$<\widehat{t}_{geno-therm}$
$\widehat{t}^{>}_{geno-therm}$
0
$\widehat{t}^{d>}_{geno-therm}$
$<\widehat{t}^{d}_{geno-therm}$
genothermodynamic motion forward against the dictates of the second law of thermodynamics
genothermodynamic motion backward against the dictates of the second law of thermodynamics

Figure 12: A schematic view of the four sets of possible "Genothermodynamic time's directions".

To incorporate in the present genothermodynamic description the above facts we need to revise appropriately the manipulation of Clausius' inequality. Therefore, the Clausius uncompensation function, $N^{>}$, is now considered as the sum of two terms $(N^{>})^{\alpha} > 0$ and $-(N^{>})^{\beta} = (N^{d>})^{\beta} > 0$, the former sums the contribution from all processes following the dictates of the second law of thermodynamics and the latter accommodates all processes proceeding against the dictates of the second law of thermodynamics. Thus we now have,

$$N^{>} = (N^{>})^{\alpha} - (N^{d>})^{\beta} = -\oint^{>} \left[\frac{1^{>}}{T^{>}} > \frac{d^{>}Q^{>}}{d^{>}t^{>}}\right] > d^{>}t^{>} > 0 \qquad (358)$$

Now instead of eq.(357) we have,

$$\begin{aligned} (N^{>})^{\alpha} &= \oint^{>} \left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\alpha} > d^{>}t^{>} > 0 \\ -(N^{>})^{\beta} = (N^{d>})^{\beta} &= \oint^{d>} \left(\frac{d^{d>}N^{d>}}{d^{d>}t^{d>}}_{d}\right)^{\beta} >^{d} d^{d>}t^{d>} > 0 \end{aligned} \qquad (359)$$

Notice that the positive sign of $(N^{d>})^{\beta}$ is because it is measured with negative units $I^{d>}$ and $I_t^{d>}$.

Thus in this way we have,

$$\begin{aligned}\frac{d^{>}N^{>}}{d^{>}t^{>}} &= \left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\alpha} + \left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\beta} \\ &= \left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\alpha} - \left(\frac{d^{d>}N^{d>}}{d^{d>}t^{d>}}d\right)^{\beta} > 0 \end{aligned} \tag{360}$$

and hence the rate of genoentropy production instead of eq.(355) reads as,

$$\frac{d_i^{>}S^{>}}{d^{>}t^{>}} = \left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\alpha} - \left(\frac{d_i^{d>}S^{d>}}{d^{d>}t^{d>}}d\right)^{\beta} > 0 \tag{361}$$

where we have the following numerical equivalence,

$$\begin{aligned}\left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\alpha} &= \left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\alpha} > 0 \\ \left(\frac{d_i^{d>}S^{d>}}{d^{d>}t^{d>}}d\right)^{\beta} &= \left(\frac{d^{d>}N^{d>}}{d^{d>}t^{d>}}d\right)^{\beta} \\ &= -\left(\frac{d^{>}N^{>}}{d^{>}t^{>}}\right)^{\beta} = -\left(\frac{d_i^{>}S^{>}}{d^{>}t^{>}}\right)^{\beta} > 0 \end{aligned} \tag{362}$$

We need to keep in mind that $(N^{d>})^{\beta}$ is not the same function the one that one may obtain on an isodual operation on $(N^{>})^{\alpha}$. The two uncompensation funtions take care of two different types of processes. The one requires the *forward genothermodynamic space* for its description and the other requires the *forward isodual-genothermodynamic space*. Similarly, the second law of thermodynamics also permits a chemical reaction proceeding at a finite rate when its chemical affinity and rate of reaction both are negative because the product of the two is positive that means its contribution to entropy production is positive. Corresponding coupled chemical reactions will also be allowed by the second law of thermodynamics. Therefore, in terms of genothermodynamics they need to be described in *backward genothermodynamic space* and *backward isodual-genothermodynamic space*.

The above facts get described as follows,

$$\frac{{}^{<}d_i\,{}^{<}S}{{}^{<}d\,{}^{<}t} = \left(\frac{{}^{<}d_i\,{}^{<}S}{{}^{<}d\,{}^{<}t}\right)^{\alpha'} - \left(\frac{{}^{<}d_i^{d}\,{}^{<}S^{d}}{{}^{<}d^{d}\,{}^{<}t^{d}}d\right)^{\beta'} > 0 \tag{363}$$

notice that we have used the different superscripts α' and β' in eq.(363) in order not to confuse with those used in the above eqs. (358) - (362), and

$$\left(\frac{<d_i<S}{<d<t}\right)^{\alpha'} = \left(\frac{<d<N}{<d<t}\right)^{\alpha'} > 0, \tag{364}$$

$$\left(\frac{<d_i^d<S^d}{<d^d<t^d}d\right)^{\beta'} = \left(\frac{<d^d<N^d}{<d^d<t^d}d\right)^{\beta'} > 0. \tag{365}$$

Now in terms of *geno-chemical affinities* and *geno-rates* of chemical reactions our expressions read as,

$$\left(\frac{d_i^> S^>}{d^> t^>}\right)^{\alpha} = \sum_{\gamma} \frac{((\mathcal{A}^>)^\gamma)^\alpha}{T^>} > \left(\frac{d^>(\xi^>)^\gamma}{d^> t^>}\right)^{\alpha} > 0, \tag{366}$$

$$\left(\frac{d_i^{d>} S^{d>}}{d^{d>} t^{d>}}d\right)^{\beta} = \sum_{\gamma} \left(\frac{\left((\mathcal{A}^{d>})^\gamma\right)^\beta}{T^{d>}}d\right) >^d \left(\frac{d^{d>}(\xi^{d>})^\gamma}{d^{d>} t^{d>}}d\right)^{\beta} > 0, \tag{367}$$

$$\left(\frac{<d_i^< S}{<d<t}\right)^{\alpha'} = \sum_{\gamma} \frac{((<\mathcal{A})^\gamma)^{\alpha'}}{<T} < \left(\frac{<d(<\xi)^\gamma}{<d<t}\right)^{\alpha'} > 0 \tag{368}$$

$$\left(\frac{<d_i^{d<} S^d}{<d^d<t^d}d\right)^{\beta'} = \sum_{\gamma} \left(\frac{\left((<\mathcal{A}^d)^\gamma\right)^\beta}{<T^d}d\right) <^d \left(\frac{<d^d(<\xi^d)^\gamma}{<d^d<t^d}d\right)^{\beta} > 0, \tag{369}$$

One may wonder that if all expressions of genoentropy production of eqs.(366) - (369) have positive sign then what superiority lies in the adoption of *four genothermodynamic spaces*. The clarification of this query as well as certain aspects of above equations are given below,

1. When eqs.(366) and (368) are projected on the traditional Euclidean space their contribution to the rate of entropy production would be positive and that of eqs.(367) and (369) would be negative. This fact establishes the crucial role of coupling of chemical reactions.

2. Eqs.(366) - (369) are the descriptions as seen by the observers of the respective genothermodynamic spaces. Those observers individually would find that the chemical reaction is proceeding strictly in accordance with the second law of thermodynamics. This is the advantage over the the traditional irreversible thermodynamic description.

3. This description of coupled chemical reactions in our opinion shows a possible way to tackle the complex processes of life sciences. Particularly the identified four genothermodynamic time's arrows in this paper, may be further transformed appropriately to rationally describe the biological processes.

Most importantly the identification of four geno-nonequilibrium thermodynamic spaces has advantage in that one is at will to choose the appropriate geno- and genoisudual- units to suit the observed fastness of the respective processes with their established independence of the evolution.

It is amusing to notice that – so far we have not incorporated the first law of thermodynamics in our description but still we are able to present the rational geno-nonequilibrium thermodynamic description of irreversible processes of spatially uniform systems.

The only roughness in the above description is that the genoentropy production equation has two types of terms one is in terms of forward genothermodynamic motion and the other is in terms of the forward isodual-genothermodynamic motion. Instead of considering it as a drawback one needs to consider that nature is compelling us to live with it. Therefore, we stress that — this aspect of the nature, so far, remained submerged that the present author for the first time surfaced it out [6] and that we have reproduced herein.

8 Concluding Remarks

The subject matter of this paper is basically addressed to the researchers from chemistry and biology disciplines. Indeed, there are three excellent monographs written by Santilli himself basically addressed to researchers from biology discipline [13,20] and to chemists [4]. Still the present author attempted in this paper to bring out a more elementary level of description of the subject matter keeping the abstract contents as minimum as possible. It is, therefore, hoped that this paper would serve well to the beginners and would encourage them to take up research in this exciting and new field of science.

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